Discrete-Time Signal Processing

Lecture 7

Frequency Response Magnitude

Frequency Response Magnitude

$$\left|H(e^{j\omega})\right|^2 =$$

$$H(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}\left(1 - z_k z^{-1}\right)}{\prod_{k=1}^{N}\left(1 - p_k z^{-1}\right)}$$

$$H^*(1/z^*) = \left[\frac{b_0}{a_0} \frac{\prod_{k=1}^{M}\left(1 - z_k z^*\right)}{\prod_{k=1}^{N}\left(1 - p_k z^*\right)}\right]^* = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}\left(1 - z_k^* z\right)}{\prod_{k=1}^{N}\left(1 - p_k^* z\right)}$$

Frequency Response Magnitude

$$C(z) = H(z) \cdot H^*(1/z^*) = \left|\frac{b_0}{a_0}\right|^2 \frac{\prod_{k=1}^{M}\left(1 - z_k z^{-1}\right)\left(1 - z_k^* z\right)}{\prod_{k=1}^{N}\left(1 - p_k z^{-1}\right)\left(1 - p_k^* z\right)}$$

- Note that poles and zeros occur in conjugate reciprocal pairs in $C(z) \rightarrow$ 1 element of pair is associated with $H(z)$ and 1 element is associated with $H^*(1/z^*)$
- Conjugate reciprocal $\Rightarrow$ if 1 element is inside the unit circle, the other element is outside the unit circle
- If $H(z)$ corresponds to a causal stable system $\Rightarrow$ all poles must be inside the unit circle

$\therefore$ Can identify poles of $H(z)$ from $C(z)$; however zeros are not uniquely specified

Frequency Response Magnitude

Example:

- If $C(z)$ is causal and stable $\Rightarrow$

- $H(z)$ can have (z_1, z_3), (z_1, z_4), (z_2, z_3), or (z_2, z_4) as zeros

Review of FR Magnitude

- Given $\left|H(e^{j\omega})\right|^2$, we can reconstruct $C(z) = H(z)\cdot H^*(1/z^*)$

$$C(z)\big|_{z=e^{j\omega}} = \left|H(e^{j\omega})\right|^2$$

- From $C(z)$, we want to determine as much as possible about $H(z)$
  - For each pole p_k of $H(z)$, $C(z)$ has a pole at p_k and $\left(p_k^*\right)^{-1}$
  - Thus if $p_k = re^{j\theta}$, then $\left(p_k^*\right)^{-1} = \frac{1}{r}e^{j\theta}$
  - If $H(z)$ is stable and causal $\Rightarrow$ pole inside the unit circle corresponds to $H(z)$ and conjugate reciprocal pole outside the unit circle corresponds to $H^*(1/z^*)$
  - Can uniquely identify poles but not zeros from $C(z)$ – different choice of zeros will give rise to filters with the same frequency response squared magnitude (within a constant scaling factor) but different phase characteristics

Review of FR Magnitude

- Suppose $H(z) = H_1(z)\left[\frac{z^{-1} - a^*}{1 - az^{-1}}\right]$

$$\Rightarrow \qquad C(z) = H(z) \cdot H^*(1/z^*)$$

$$=$$

- Term $\left[\frac{z^{-1} - a^*}{1 - az^{-1}}\right]$ drops out from $C(z)$ via pole-zero cancellation
- This is called an allpass term and does not affect the magnitude of the frequency response of the filter

Allpass Systems

- Systems that have constant magnitude are called 'allpass' systems

$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

- Zero at $z = 1/a^* = \left(1/r\right)e^{j\theta}$
- Pole at $z = a = re^{j\theta}$

$$\left|H_{ap}(e^{j\omega})\right|^2 = \left(\frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}}\right)\left(\frac{e^{j\omega} - a}{1 - a^*e^{j\omega}}\right)$$

$$= \frac{1 - ae^{-j\omega} - a^*e^{j\omega} + |a|^2}{1 - ae^{-j\omega} - a^*e^{j\omega} + |a|^2} = 1, \quad \forall \omega$$

- Allpass systems pass all frequencies with constant gain

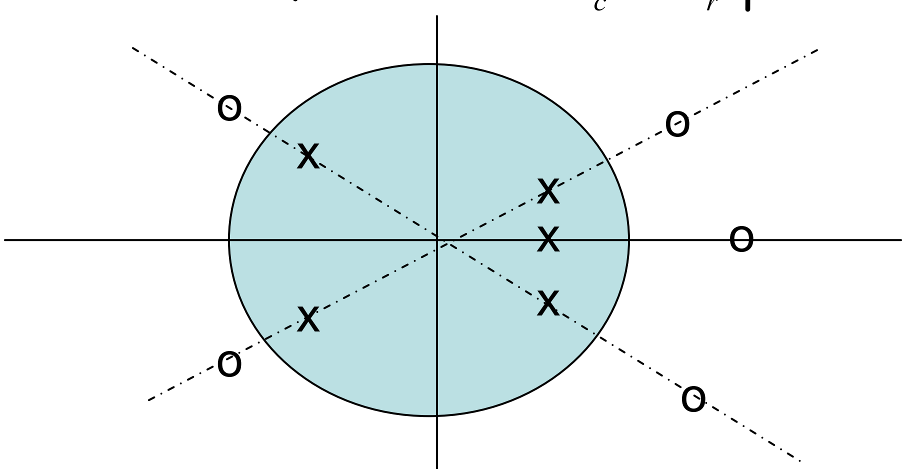
Allpass Systems

- If $H(z)$ corresponds to a real-valued allpass system $\Rightarrow$ all poles and zeros occur in complex-conjugate pairs, therefore:

$$H(z) = A\prod_{k=1}^{M_r}\frac{z^{-1}-d_k}{1-d_k z^{-1}}\prod_{k=1}^{M_c}\frac{\left(z^{-1}-e_k^*\right)\left(z^{-1}-e_k\right)}{\left(1-e_k z^{-1}\right)\left(1-e_k^* z^{-1}\right)}$$

where the d_k's are real poles/zeros and the e_k's are complex conjugate poles/zeros (with $|d_k|<1, |e_k|<1$ for a stable, causal system)

- There are a total of $M = N = 2M_c + M_r$ poles or zeros



Allpass Systems

- Each pole in an allpass system is paired with a conjugate reciprocal zero
- For causal, stable systems, poles are located inside the unit circle and zeros are located at conjugate reciprocal locations outside the unit circle

$$H_{AP}(e^{j\omega}) = \frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}$$

$$\left|H_{AP}(e^{j\omega})\right|^2 = 1$$

$$\measuredangle H_{AP}(e^{j\omega}) = -\omega - 2\tan^{-1}\left[\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}\right]$$

$$grd\left\{H_{AP}(e^{j\omega})\right\} = \frac{1-r^2}{\left|1 - re^{j\theta}e^{-j\omega}\right|^2}$$

- Causal, stable AP system $\Rightarrow$ $|r| < 1, \therefore grd\left\{H_{AP}(e^{j\omega})\right\} > 0 \;\; \forall\omega$
- Group Delay of higher order systems $=$ sum of grd of first order systems

  $\Rightarrow$ $grd \; > 0 \; \forall\omega$ for all causal, stable AP systems

Frequency Response Notes

- Notes:
  - The frequency response magnitude does not uniquely specify $H(z)$
  - Allpass terms cannot be deduced from $C(z)$
  - Stable and causal systems $\Rightarrow$ poles must be inside the unit circle but it does not specify the location of the zeros
  - Inverse systems: $H_i(z)$ is the inverse of $H(z)$ if $G(z) = H(z)H_i(z) = 1$

$$\Rightarrow \; H_i(z) = \frac{1}{H(z)} \quad \text{or} \quad H_i(e^{j\omega}) = \frac{1}{H(e^{j\omega})}$$

$$\therefore \quad h(n) * h_i(n) = \delta(n)$$

- Not all systems have inverses! e.g., ideal LP filter

Minimum Phase Systems

- A minimum phase system is one that is stable and causal with a stable and causal inverse $\Rightarrow$ all poles and zeros of $H(z)$ must be inside the unit circle
  - Zeros of $H(z)$ become poles of $H_i(z)$
  - Need all poles of $H(z)$ and $H_i(z)$ to be inside the unit circle
- Any rational system function can be written as:

$$H(z) = H_{MP}(z) H_{AP}(z)$$

- Poles (zeros) outside the unit circle are reflected (conjugate reciprocal) inside the unit circle--with pole for allpass term and to cancel the reflected zero

Minimum Phase Systems

- Proof that $H(z) = H_{MP}(z)H_{AP}(z)$
  - $H_{MP}(z)$ has all poles and zeros inside the unit circle
  - If $H(z)$ has a zero outside the unit circle at $1/c^*$, we can write $H(z)$ in the form:

  where $|c| < 1$ and $H_1(z)$ is a minimum phase system.

  Thus we can write

  where $\frac{z^{-1} - c^*}{(1 - cz^{-1})}$ is an allpass term. $H_2(z)$ is MP since

  $H_1(z)$ is MP with a zero at $z = c$ added, where $|c| < 1;$ can do similar analysis for all poles and zeros outside the unit circle

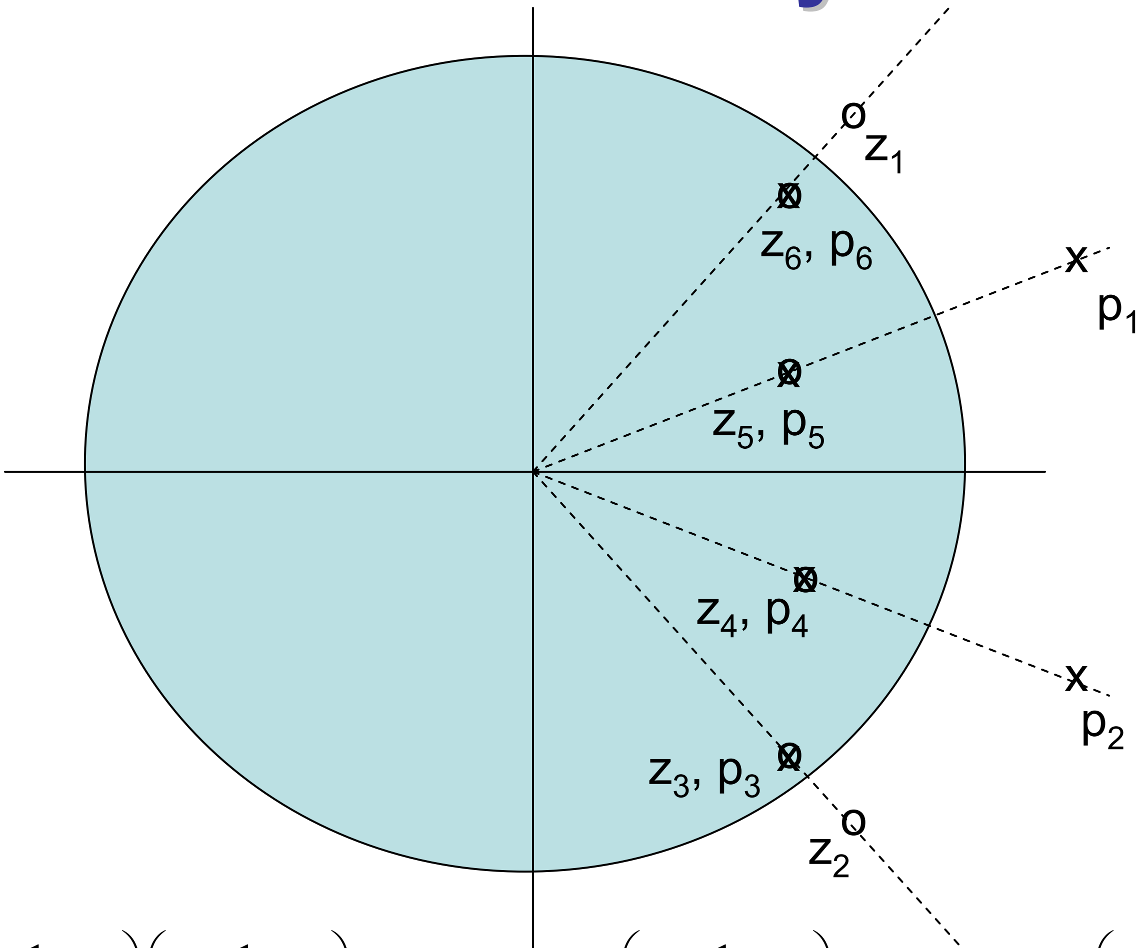
  $\therefore \; H(z) = H_{MP}(z)H_{AP}(z)$
- Zeros (poles) outside the unit circle are reflected to conjugate reciprocal locations inside the unit circle with a corresponding pole for the allpass term

Minimum Phase Systems

- Example:

$$H(z) = \frac{1 - 2z^{-1}}{1 - (1/3)z^{-1}}$$

Minimum Phase Systems



$$H(z)=\frac{\left(1-\frac{1}{z_1^*}z^{-1}\right)\left(1-\frac{1}{z_2^*}z^{-1}\right)}{\left(1-\frac{1}{p_1^*}z^{-1}\right)\left(1-\frac{1}{p_2^*}z^{-1}\right)}\cdot\frac{\left(1-z_1z^{-1}\right)}{\left(1-\frac{1}{z_1^*}z^{-1}\right)}\cdot\frac{\left(1-\frac{1}{p_1^*}z^{-1}\right)}{\left(1-p_1z^{-1}\right)}\cdot\frac{\left(1-z_2z^{-1}\right)}{\left(1-\frac{1}{z_2^*}z^{-1}\right)}\cdot\frac{\left(1-\frac{1}{p_2^*}z^{-1}\right)}{\left(1-p_2z^{-1}\right)}$$

$$= \qquad M.P. \qquad \cdot \qquad A.P.$$

Minimum Phase Systems

- Properties of M.P. Systems:

$$H(e^{j\omega}) = H_{MP}(e^{j\omega}) \cdot H_{AP}(e^{j\omega})$$

$$\left|H(e^{j\omega})\right| = \left|H_{MP}(e^{j\omega})\right|$$

1. $$\arg\left\{H(e^{j\omega})\right\} = \arg\left\{H_{MP}(e^{j\omega})\right\} + \arg\left\{H_{AP}(e^{j\omega})\right\}$$

$\arg\left\{H(e^{j\omega})\right\} < \arg\left\{H_{MP}(e^{j\omega})\right\}$ (because $\arg\left\{H_{AP}(e^{j\omega})\right\} < 0$ since the group delay is greater than 0 for A.P. Systems)

$\Rightarrow$ M.P. Systems have minimum <u>phase lag</u> (the negative of phase)

2. $$grd\left\{H(e^{j\omega})\right\} = grd\left\{H_{MP}(e^{j\omega})\right\} + grd\left\{H_{AP}(e^{j\omega})\right\}$$

– For causal, stable systems $grd\left\{H_{AP}(e^{j\omega})\right\} > 0 \;\forall \omega$

$$\Rightarrow grd\left\{H(e^{j\omega})\right\} \geq grd\left\{H_{MP}(e^{j\omega})\right\}$$

Minimum Phase Systems

- Properties of M.P. Systems:

$$H(e^{j\omega}) = H_{MP}(e^{j\omega}) \cdot H_{AP}(e^{j\omega})$$

$$\left|H(e^{j\omega})\right| = \left|H_{MP}(e^{j\omega})\right|$$

3. $$\sum_{m=0}^{n} \left|h(m)\right|^2 \leq \sum_{m=0}^{n} \left|h_{MP}(m)\right|^2 \qquad \forall n$$

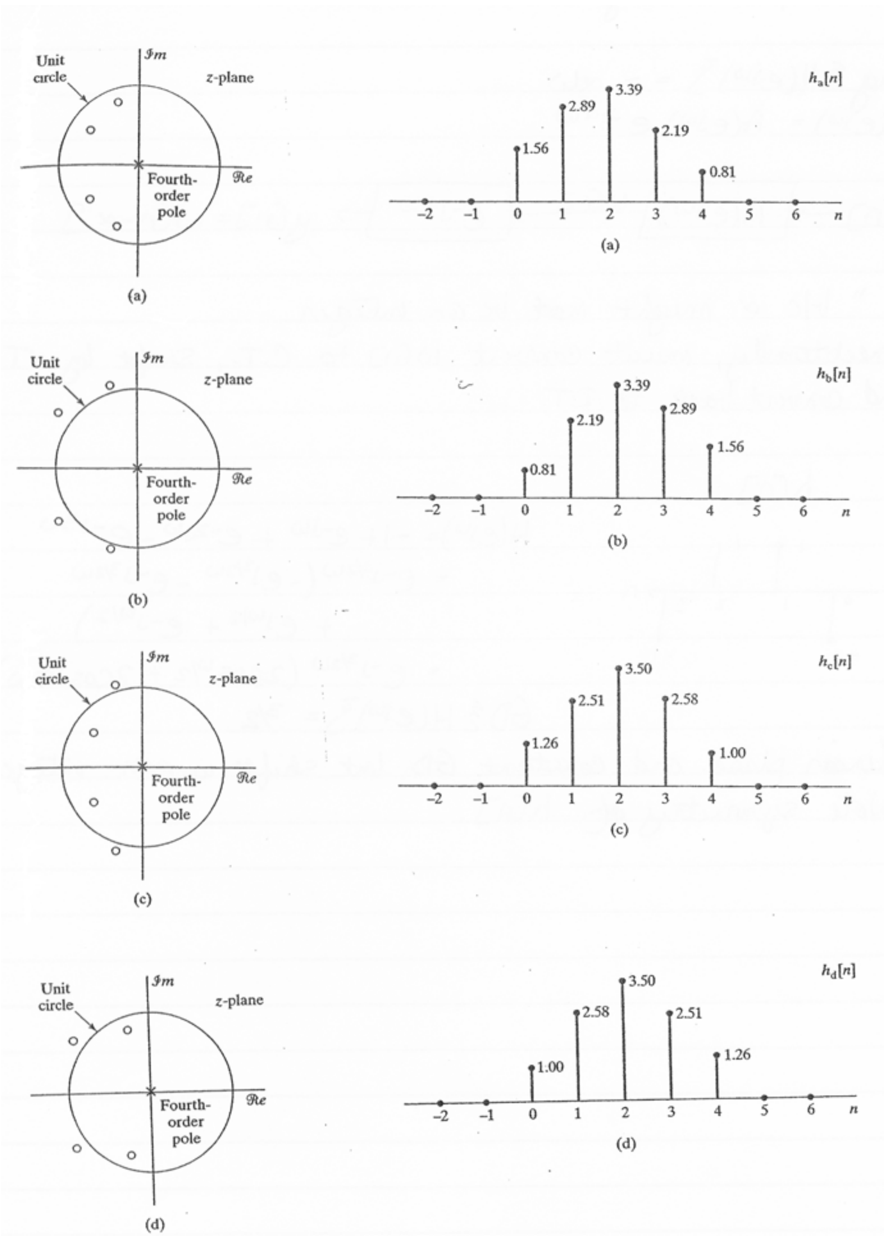
- This is the "Minimum energy delay property" whereby M.P. Systems have energy that is closest to the beginning of the sequence $\Rightarrow \quad \left|h(0)\right| \leq \left|h_{MP}(0)\right|$

- M.P. systems get their energy out the fastest...

Note: Maximum Phase systems have all their zeros outside the unit circle and get their energy out the slowest

$$\sum_{m=0}^{n} \left|h(m)\right|^2 \geq \sum_{m=0}^{n} \left|h_{MaxPh}(m)\right|^2 \qquad \forall n$$

Minimum Phase Systems



Four systems all having the same frequency response magnitude. Zeros are at all combinations of $0.9e^{\pm j0.6\pi}$ and $0.8e^{\pm j0.8\pi}$ and their reciprocals, and their corresponding sequences.