Discrete-Time Signal Processing

Lectures 5-6

Analysis of LTI Systems

Analysis of LTI Systems

- Discrete-Time LTI System Characterization

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) = x * h$$

$$Y(z) = X(z)H(z), \quad R_y \supset R_x \cap R_h$$

$h(n) =$ impuse response of system

$H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n} =$ system function

Analysis of LTI Systems

Example:

- Assume $|a| < 1$, then the ROC for $Y(z)$ is $|z| > 1$
- Using Partial Fraction Expansion, can express $Y(z)$ as:
- This is the unit step response of the filter h

Frequency Response

- Suppose input to a system is a complex sinusoid of the form:

$$x(n) = e^{j\omega_0 n}, \; -\infty < n < \infty$$

$$\Rightarrow \qquad y(n) = \sum_{k=-\infty}^{\infty} h(k) e^{j\omega_0 (n-k)}$$

$$= e^{j\omega_0 n} \sum_{k=-\infty}^{\infty} h(k) e^{-j\omega_0 k}$$

$$= e^{j\omega_0 n} H(z)\Big|_{z=e^{j\omega_0}}$$

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$$ assuming it exists (ROC includes unit circle in z-plane)

Frequency Response

- The Frequency Response of the LTI system, $H(e^{j\omega})$, is the response of the system to a complex sinusoid, $e^{j\omega n}$

$$e^{j\omega n} \rightarrow \boxed{\text{LTI System}} \rightarrow$$

- By linearity, if $x(n) = \sum_k a_k e^{j\omega_k n} \rightarrow \boxed{\text{LTI System}} \rightarrow$
- The frequency response dictates how each frequency present in the input signal is altered when passed through the system. The system alters both the amplitude and the phase.
- Exponentials are eigenfunctions of LTI systems.

Frequency Response

- Frequency Response Representations:

$$x(n) \rightarrow \boxed{\text{LTI System}} \rightarrow y(n)$$

$$Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega})$$

$$\left|Y(e^{j\omega})\right| = \left|X(e^{j\omega})\right| \cdot \left|H(e^{j\omega})\right|$$

$$\measuredangle Y(e^{j\omega}) = \measuredangle X(e^{j\omega}) + \measuredangle H(e^{j\omega})$$

- Continuous Time: $\Omega : -\infty \rightarrow \infty$
- Discrete Time: ω: any 2π interval, e.g., $0 \rightarrow 2\pi$, or $-\pi \rightarrow \pi$
- $e^{j\omega}$ is periodic with period $2\pi \Rightarrow H(e^{j\omega})$ is also periodic with period 2π

Frequency Response

- Properties of Frequency Response:

1. DC response

2. High frequency response

Frequency response corresponds to evaluating the system response along the unit circle in the z-plane; can also see 2π periodicity in ω

Frequency Response

3. Symmetry Properties:

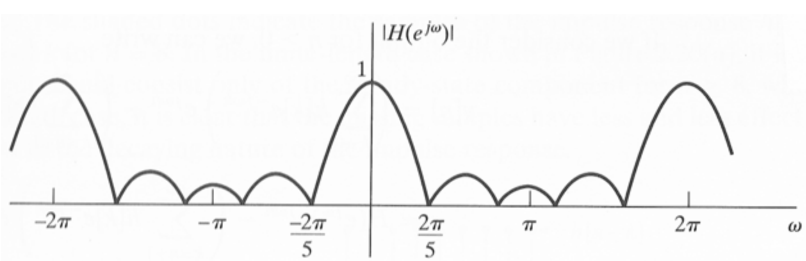
If $h(n)$ real, $H^*(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n)e^{j\omega n}$

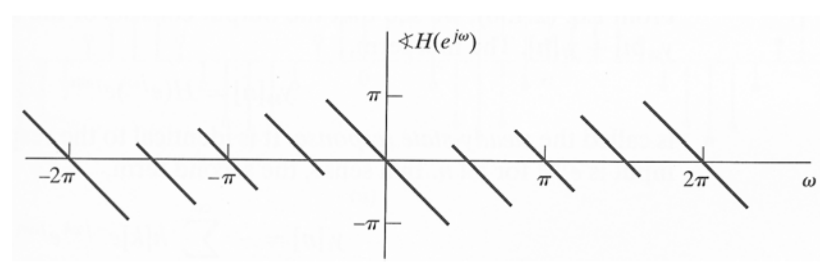
$$= \sum_{n=-\infty}^{\infty} h(n)e^{-j(-\omega)n} = H(e^{-j\omega})$$

$$\therefore H(e^{j\omega}) = H^*(e^{-j\omega})$$

$\Rightarrow \left|H(e^{j\omega})\right| = \left|H(e^{-j\omega})\right|$ - magnitude is even function of ω

$\measuredangle H(e^{j\omega}) = -\measuredangle H(e^{-j\omega})$ - phase is odd function of ω





Frequency Response

4. Stability Properties:

$$\sum_{n=-\infty}^{\infty} \left|h(n)\right| < \infty$$

$$\Rightarrow \quad \sum_{n=-\infty}^{\infty} \left|h(n)\right|z^{-n}\Big|_{z=1} < \infty$$

$$\Rightarrow \quad \left|\sum_{n=-\infty}^{\infty} h(n)z^{-n}\right|\Bigg|_{z=1} < \infty$$

$$\Rightarrow \quad H(z)\Big|_{z=1} < \infty$$

$\therefore \quad R_h$ must contain the unit circle in the z-plane ($|z| = 1$)

for system to be stable $\Rightarrow H(e^{j\omega})$ exists

Phase Representations

Principle Value of Phase: $-\pi < ARG\left\{H(e^{j\omega})\right\} \leq \pi$

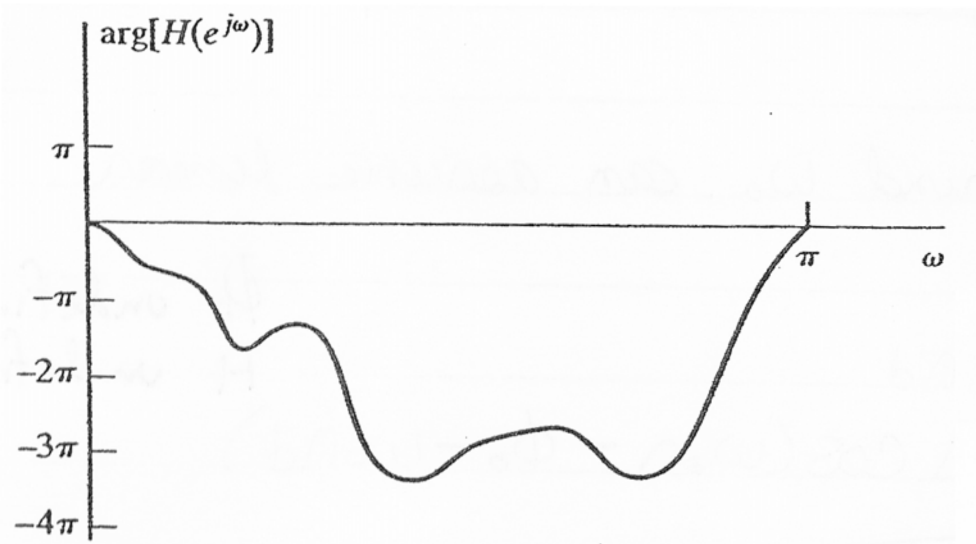
$$\measuredangle H(e^{j\omega}) = ARG\left\{H(e^{j\omega})\right\} + 2\pi r$$

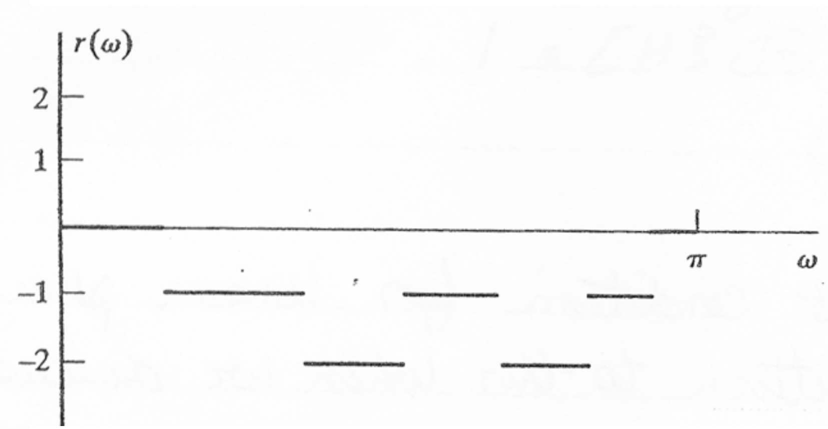
ARG does not commute with addition

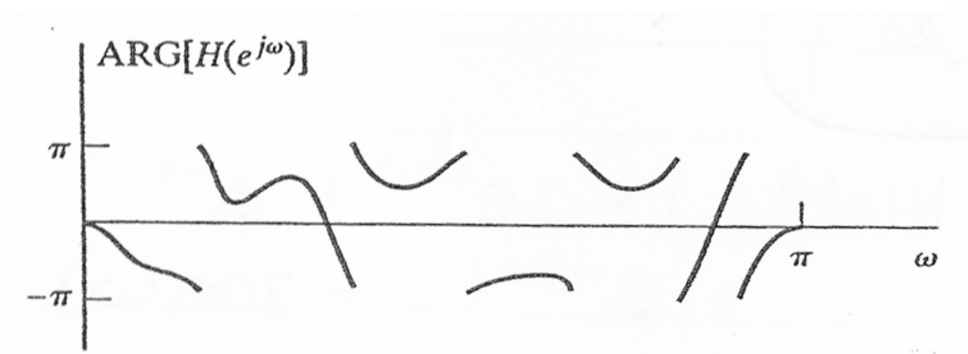
Example:

$\arg\left\{H(e^{j\omega})\right\}$ is a continuous function, $-\pi < \omega \leq \pi$

Example of Phase Curves







Group Delay

$$\text{Group Delay} = \tau(\omega) = -\frac{d\left[\arg\left\{H(e^{j\omega})\right\}\right]}{d\omega}$$

- Group Delay, $\tau(\omega)$, shows the number of samples of delay at a single frequency (ω)
- Constant Group Delay $\Rightarrow$ equal delay at all frequencies, i.e., $\tau(\omega) = k$ (samples) $\Rightarrow \arg\left\{H(e^{j\omega})\right\} = -k\omega$
- Linear phase systems have constant Group Delay

Group Delay

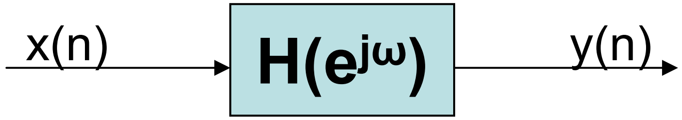
- Interpretation of Group Delay:

  Group Delay $=$ response of system to narrowband signal at frequency ω

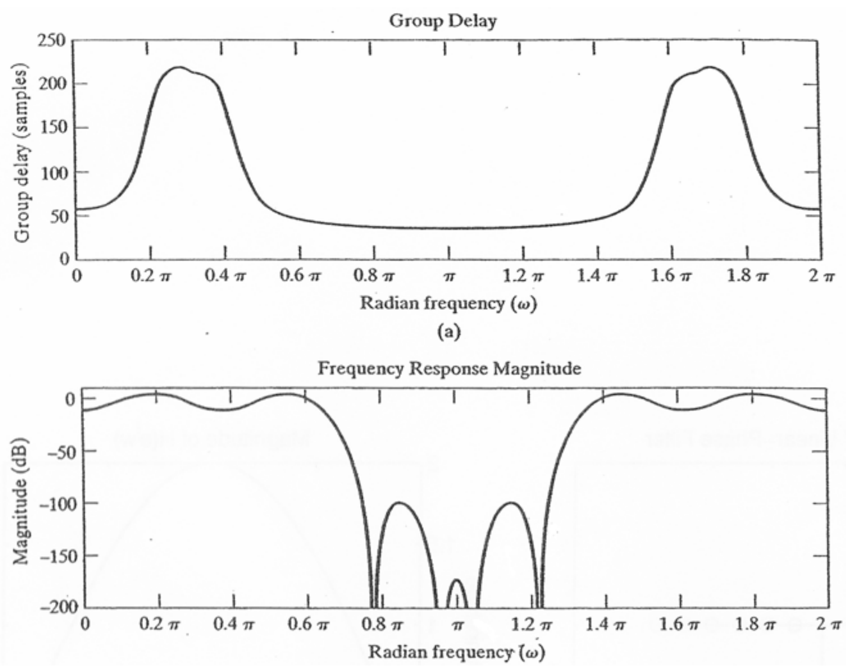
Example:

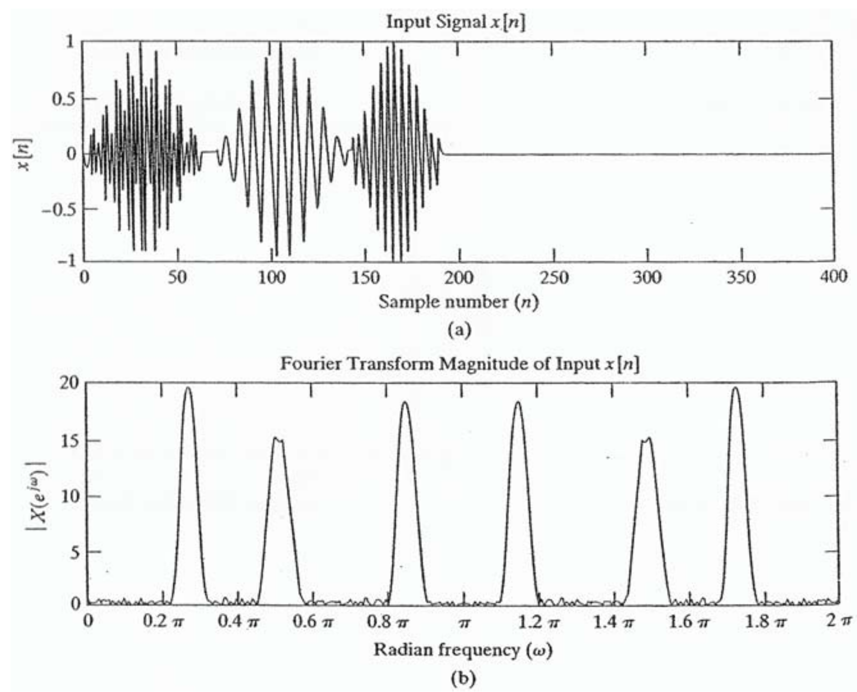
- Since $x(n)$ is narrowband around ω_0, we can show the following:

$\Rightarrow$ Group delay of n_d applied to signal $s(n)$ and phase delay of $-\phi_0 - \omega_0 n_d$ applied to carrier for narrowband signal



Example of GD

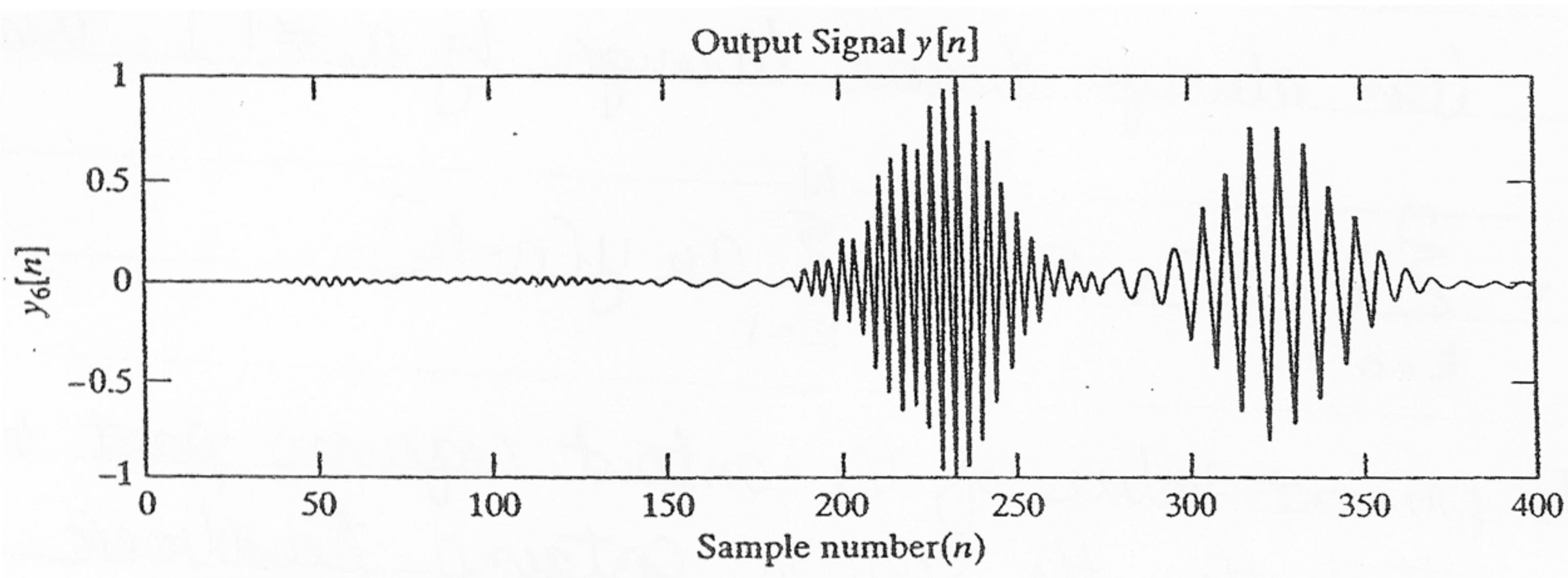




Plots of system group delay and frequency response magnitude—what type of system is this?

Plots of input signal and input signal magnitude—what happens when x(n) is processed by h(n)?

Example of GD

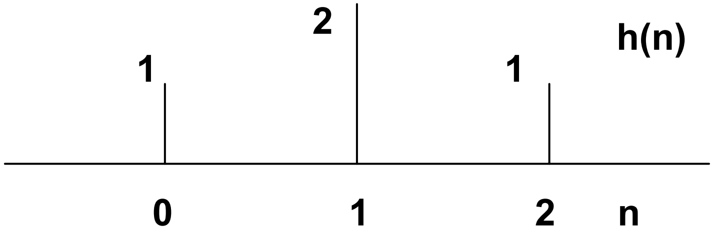


The band rejection part of the system response removes the narrowband signal around ω=0.8π; the group delay of the system delays the narrowband signal around ω=0.3π by 200 samples, and the narrowband signal around ω=0.3π by 50 samples, thereby reversing the order of the two signals in time.

Group Delay

Example:

- Symmetric $h(n)$ is a sufficient condition for linear phase and constant Group Delay. We will return to this when we discuss linear phase systems.



Difference Equations

- Recall LCCDE: $$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \qquad (1)$$
- Need auxiliary conditions to completely specify system, e.g., initial rest $\Rightarrow$ causal, LTI system
- If we arbitrarily set $a_0 = 1$ (can always divide entire equation by $a_0 \neq 1$), then we get:

$$y(n) = \sum_{k=0}^{M} b_k x(n-k) - \sum_{k=1}^{N} a_k y(n-k)$$

- Note: if the present value of the output requires past values of the output, then the system uses feedback and is called recursive. Otherwise the system is non-recursive.

Difference Equations

- It is easier to analyze LCCDEs of LTI systems in the z-domain.
- Taking $z-$transforms of the basic equation:

$\Rightarrow$ Can obtain the system function $H(z)$ directly from LCCDE and vice versa

LCCDE Example

Example #1:

$$H(z) = \frac{2 + \frac{1}{3}z^{-1}}{1 + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-3}} \qquad |z| > \frac{1}{2}$$

LCCDE Example

Example #2:

$$y(n) = \sum_{m=0}^{M} b_m x(n-m)$$

- There is no feedback--only a finite number of "taps"

$\Rightarrow$ FIR filter

LCCDE Example

Example #3:

$$y(n) = x(n) + ay(n-1)$$

- This filter has feedback
- There are an infinite number of taps $\Rightarrow$ IIR filter
- Filter can be implemented ***recursively***

Geometric Evaluation of H(z)

- Given a system of the form:

$$H(z)=\frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{k=0}^{N} a_k z^{-k}}$$

i.e., a ratio of an Mth order numerator polynomial and an Nth order denominator polynomial, we can express $H(z)$ as:

$$H(z)=\frac{b_0}{a_0}\frac{\prod_{m=1}^{M}\left(1-z_m z^{-1}\right)}{\prod_{k=1}^{N}\left(1-p_k z^{-1}\right)}=\frac{b_0 z^{N-M}}{a_0}\frac{\prod_{m=1}^{M}\left(z-z_m\right)}{\prod_{k=1}^{N}\left(z-p_k\right)}$$

- $\{z_m\}$ is the set of M zeros of $H(z)$; $\{p_k\}$ is the set of N poles of $H(z)$
- There are $N-M$ zeros at 0 if $N>M$ and $N-M$ poles at 0 if $N<M$

Geometric Evaluation of H(z)

- Each term is the expression for $H(z)$ can be written as:

- For each pole and zero there is a similar term $\Rightarrow$

$$H(z) = z^{N-M}\,\frac{b_0}{a_0}\,\frac{\prod_{m=1}^{M} B_m \cdot e^{\left(j\sum_{m=1}^{M}\theta_m\right)}}{\prod_{k=1}^{N} A_k \cdot e^{\left(j\sum_{k=1}^{N}\phi_k\right)}}$$

Geometric Evaluation of H(z)

- On the unit circle,

$\Rightarrow$ $$\left|H(e^{j\omega})\right| =$$

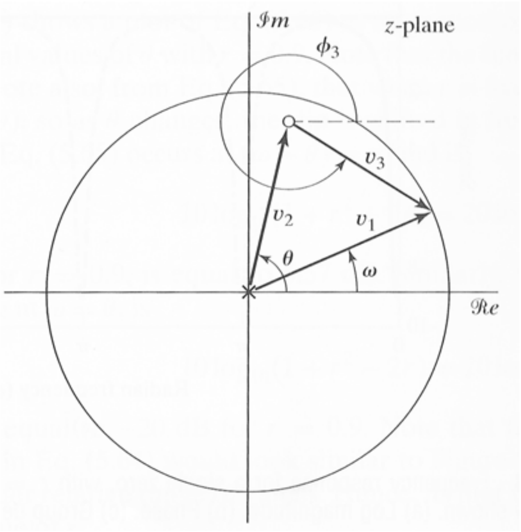
$$\measuredangle H(e^{j\omega}) =$$

- Can use these equations to evaluate $H(e^{j\omega})$ geometrically

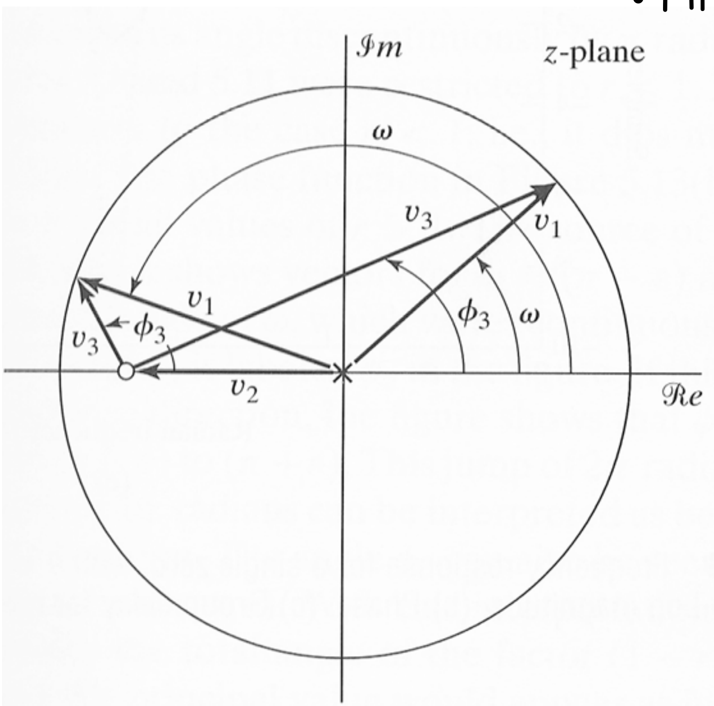
Single Zero or Pole

- Frequency response of a single zero or pole:

$$H(z) = 1 - re^{j\theta} z^{-1} = \frac{z - re^{j\theta}}{z}, \quad r < 1$$



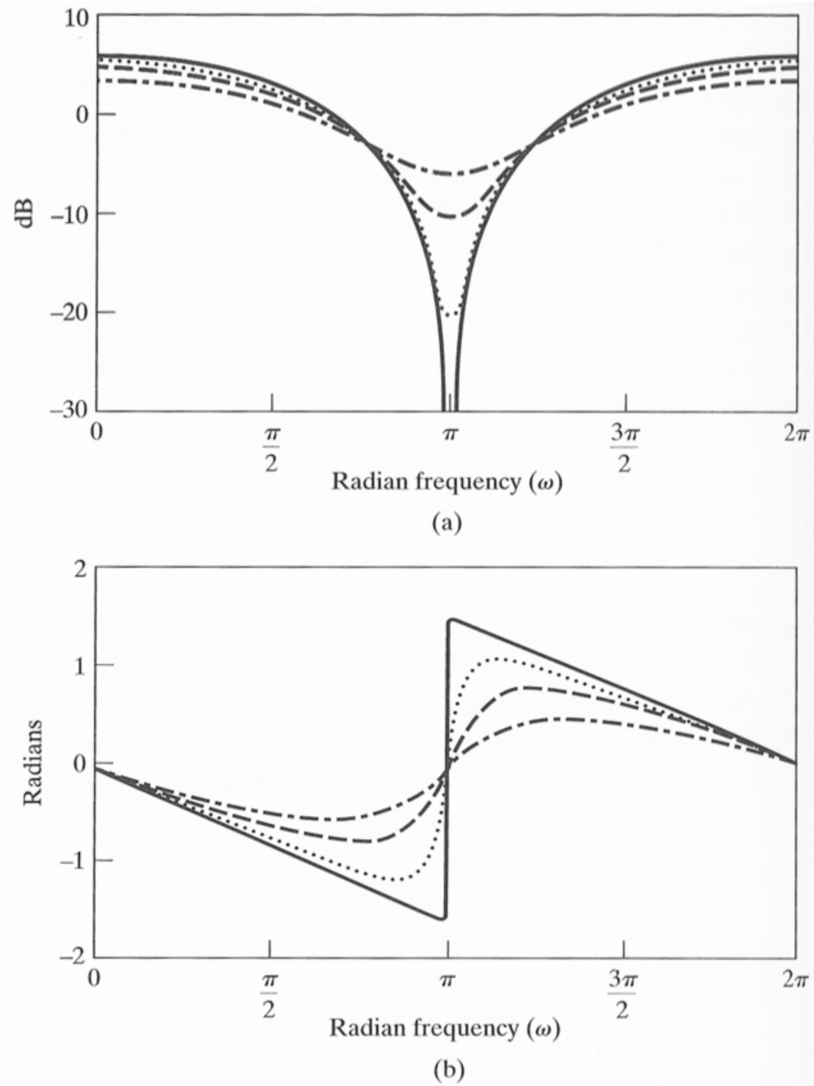
Single Zero or Pole



- First order system with $\theta = \pi$
  - $H(e^{j\omega})$ starts at maximum value at $\omega = 0$
  - $H(e^{j\omega})$ gets smaller and smaller until it achieves minimum value at $\omega = \pi$
  - $H(e^{j\omega})$ goes from minimum to maximum at $\omega = 2\pi$

- $\measuredangle H(e^{j\omega})$ beings at 0 and goes negative because ϕ increases more slowly than ω
- Near $\omega = \pi$, ϕ starts increasing faster than ω, so $\measuredangle H(e^{j\omega})$ has positive slope
- at $\omega = \pi$, ϕ is larger than ω, so $\measuredangle H(e^{j\omega})$ is positive

Single Pole Frequency Response



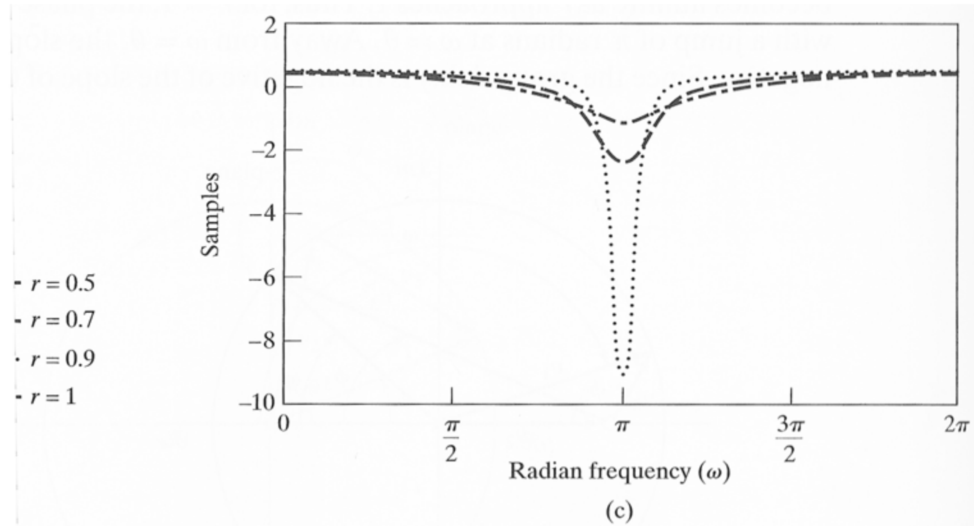


Figure 5.11 Frequency response for a single zero, with $\theta = \pi$, $r = 1$, 0.9, 0.7, and 0.5. (a) Log magnitude. (b) Phase. (c) Group delay for $r = 0.9$, 0.7, and 0.5.

Second Order System

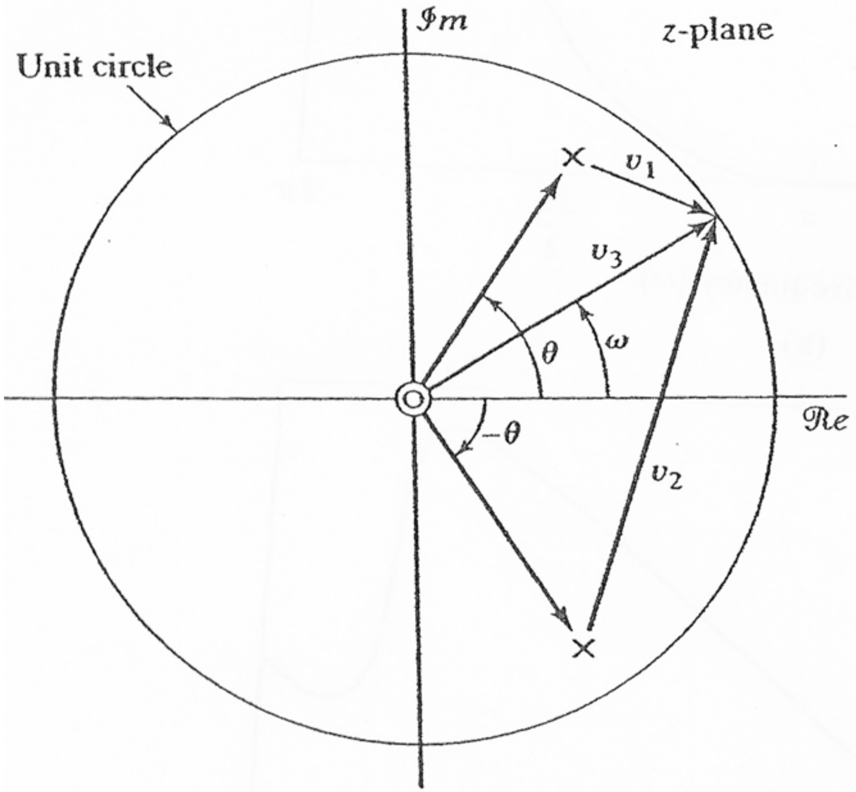
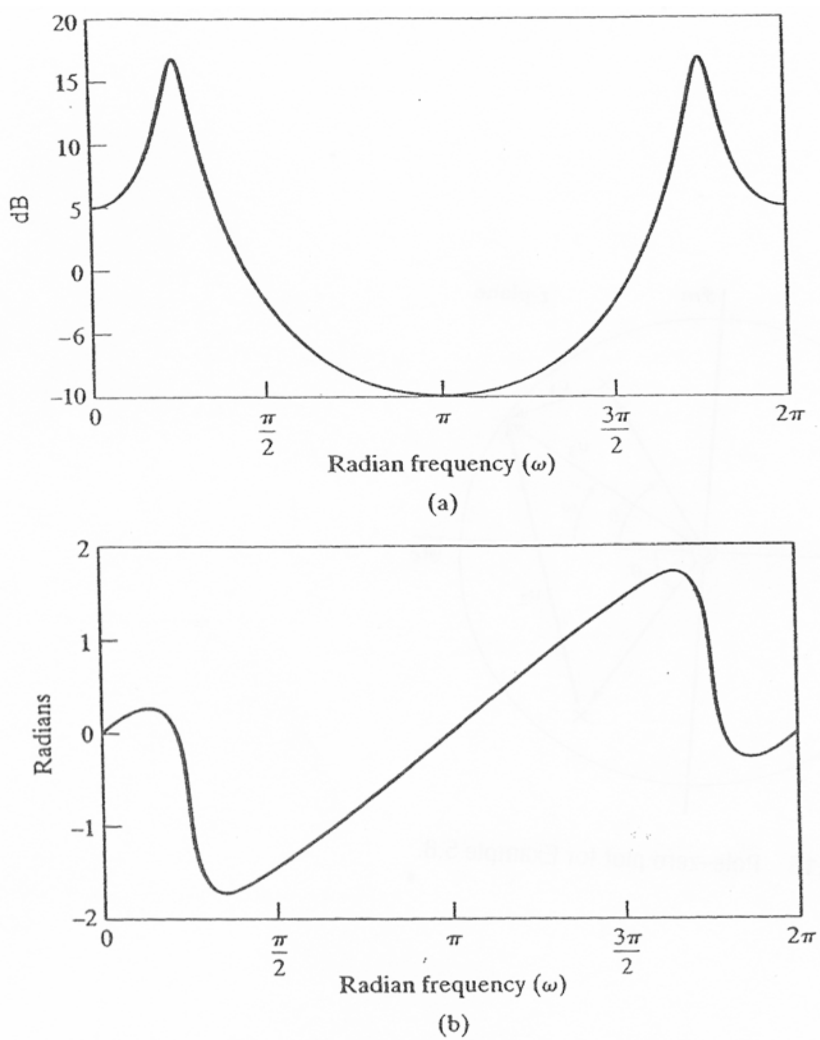


Figure 5.15 Pole–zero plot for Example 5.8.

$\left|H(e^{j\omega})\right| =$

$\left|H(e^{j\omega})\right|$ becomes large in the vicinity of the pole location (either $|v_1|$ or $|v_2|$ becomes small)

Second Order System



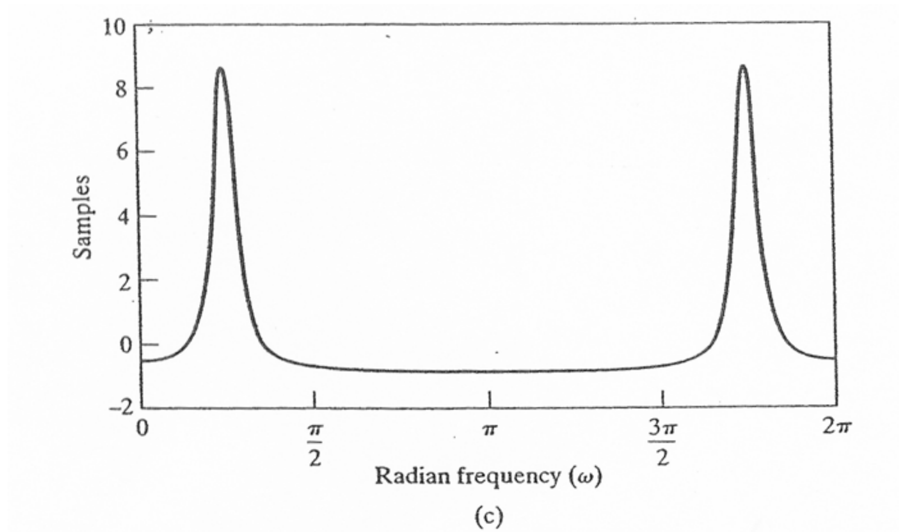


Figure 5.16 Frequency response for a complex-conjugate pair of poles as in Example 5.8, with $r = 0.9, \pi/4$. (a) Log magnitude. (b) Phase. (c) Group delay.

Example

$$y(n) = \frac{1}{M+1}\sum_{m=0}^{M} x(n-m) \qquad \text{(Averaging Filter)}$$

$$H(z) = \frac{1}{M+1}\sum_{m=0}^{M} z^{-m} = \frac{1}{M+1}\frac{\left(1-z^{-(M+1)}\right)}{\left(1-z^{-1}\right)}$$

- Zeros occur at $\left(1-z^{-(M+1)}\right)=0 \Rightarrow z^{-(M+1)} = 1 = e^{j2\pi m}$

$$z_m = e^{j2\pi m/(M+1)}, \qquad m = 1, 2, \ldots, M$$

- Note that there is no zero at $z=1$ (the $m=0$ term) because it is cancelled by the pole at $z=1$

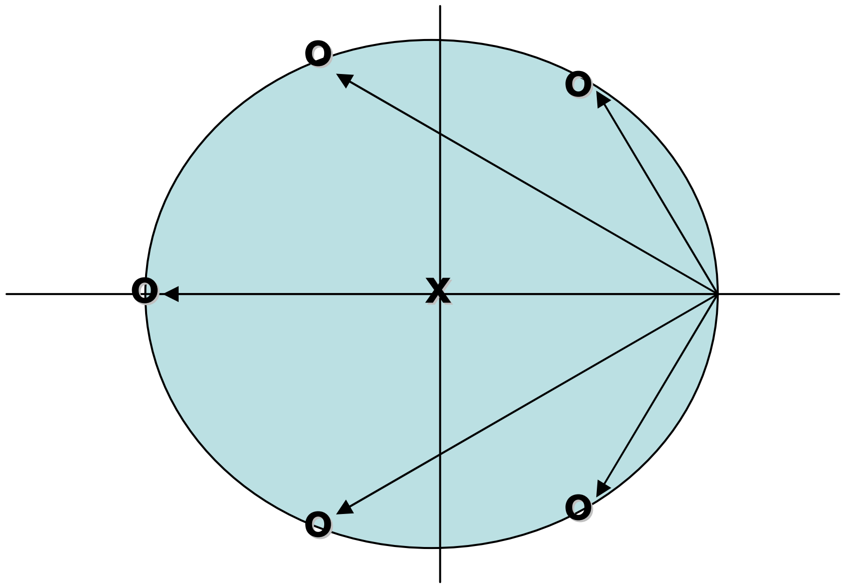
$\therefore$ There are M zeros on the unit circle at $z_m = e^{j2\pi m/(M+1)}$

Example

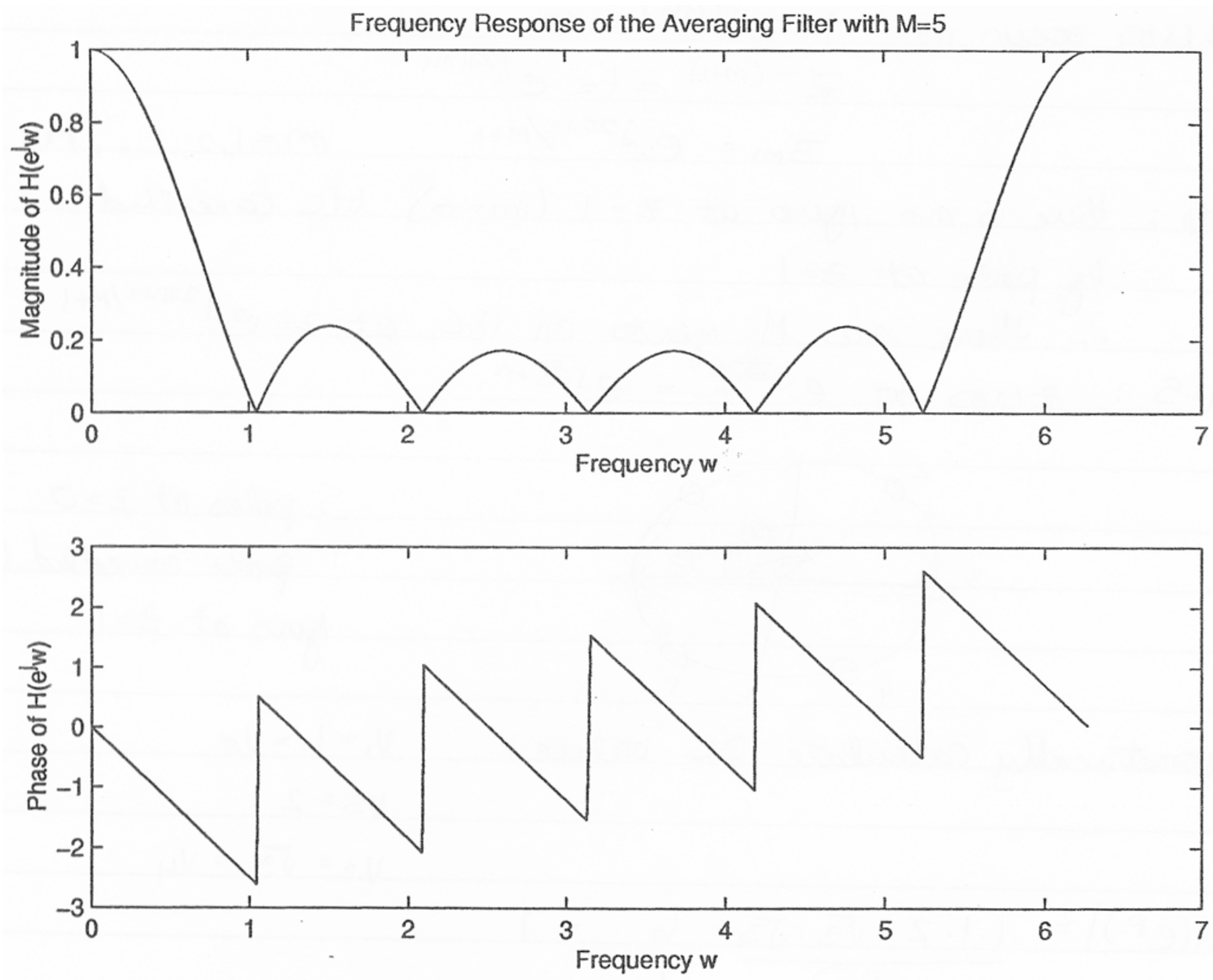
- Consider $M = 5 \Rightarrow$ zeros at $e^{j2\pi m/6} = e^{j\pi m/3}$
- There are 5 poles at $z = 0$ with the 1 pole at $z = 1$ cancelled by the zero at $z = 1$
- We can geometrically calculate the value of $H(e^{j0})$ as follows:

$$v_1 = 1 = v_5;\ v_3 = 2;\ v_2 = \sqrt{3} = v_4$$

$$\left|H(e^{j0})\right| = \frac{1 \cdot 1 \cdot 2 \cdot \sqrt{3} \cdot \sqrt{3}}{M+1} = \frac{6}{6} = 1$$



Frequency Response of Averaging Filter



Geometric Frequency Response

- Can think of zeros as "stakes" and poles as "poles" supporting a tent
  - Zeros push the tent (the frequency response) to the ground (zero)
  - Poles push the tent towards the sky (large values)
- As ω traverses around the unit circle, when close to a pole, the frequency response is large (the tent is pushed upwards); whereas when close to a zero, the frequency response is small (tent staked to the ground). If the zero (pole) is on the unit circle $\Rightarrow$ the frequency response goes to 0 (∞) at that point.