Discrete-Time Signal Processing

Lecture 4

z-Transforms

The z-Transform

- The z-transform plays the same role for Discrete-Time (DT) systems that the Laplace transform plays for Continuous Time (CT) systems

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}, \qquad z = \text{complex variable, } z \in \mathbb{R}$$

- The z-transform is bilateral (2-sided z-transform); unilateral z-transforms defined similarly except range on summation for $n \geq 0$, or $n \leq 0$

Note: $X(e^{j\omega}) = X(z)\big|_{z=e^{j\omega}}$ when it exists. Thus the discrete Fourier transform of a sequence is the z-transform evaluated on the unit circle ($|z| = 1$). This explains why the DTFT is periodic with period $\omega/(2\pi)$

Regions of Convergence (ROC)

- The Region of Convergence (ROC) of a z-transform is the region in the complex z-plane where $\left|X(z)\right| < \infty$

$$\left|X(z)\right| = \left|\sum_n x(n) z^{-n}\right| \le \sum_n \left|x(n)\right|\left|z^{-n}\right|$$

$$= \sum_n \left|x(n)\right|\left|z\right|^{-n}$$

$\therefore$ If $z = z_0$ is in the ROC $\Rightarrow$ all $\left|z\right| = \left|z_0\right|$ will be in the ROC

$\Rightarrow$ ROC contains annular regions (rings centered around $z = 0$)

- ROC generally defined by: $r_- < \left|z\right| < r_+$

Regions of Convergence

Other possible ROCs:

- Note that $z = 0$ or $z = \infty$ may or may not be contained in these regions of convergence
- Assume $X(z)$ is a rational function of the form:

$$X(z) = \frac{N(z)}{D(z)};$$

the roots of $N(z)$ are the zeros of $X(z)$ $\left(X(z_0) = 0 \right)$

the roots of $D(z)$ are the poles of $X(z)$ $\left(X(z_p) = \infty \right)$

- Since $X(z) \to \infty$ at the poles $\Rightarrow$ the locations of the poles are ***not*** in the ROC
- The ROC of $X(z)$ is actually bounded by the poles of $X(z)$
- The zeros of $X(z)$ can be either inside or outside the ROC

Examples of ROC

- Example #1: Impulse

$$x(n) = \delta(n)$$

- Example #2: Delayed Impulse

$$x(n) = \delta(n - n_d)$$

- Example #3: Unit Step

$$x(n) = u(n)$$

Examples of ROC

- Example #4: Causal Exponential

$x(n) = a^n u(n)$ (causal)

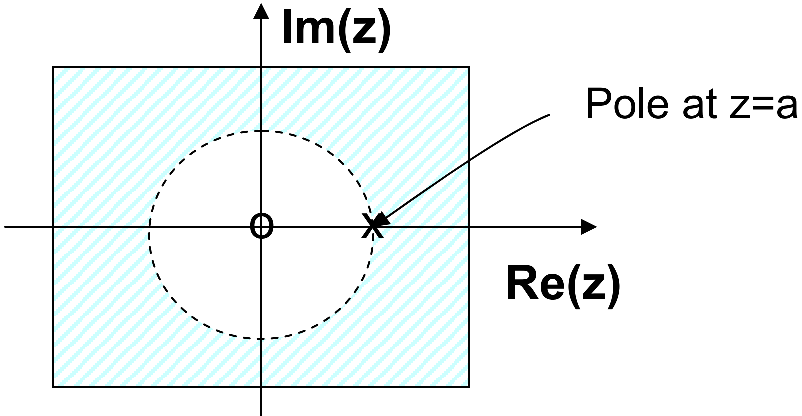
$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n = \frac{1}{1-az^{-1}}; \quad \text{ROC}: \left|az^{-1}\right| < 1 \Rightarrow |z| > |a|$$

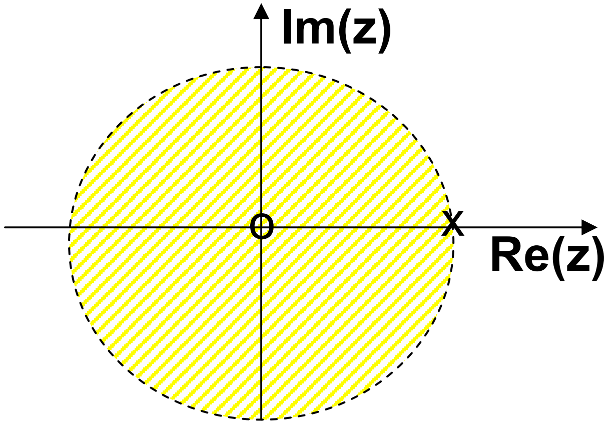
- Example #5: Non-Causal Exponential

$x(n) = -a^n u(-n-1)$ ($u(-n-1) = 1;\ -n-1 \geq 0$ or $n \leq -1;\ 0\ otherwise$)

$$X(z) = \sum_{n=-\infty}^{-1} -a^n z^{-n} = -\sum_{n=-\infty}^{-1} \left(az^{-1}\right)^n = \frac{1}{1-az^{-1}}; \quad \text{ROC}: \left|az^{-1}\right| > 1 \Rightarrow |z| < |a|$$

- Note: the functional form of the z-transforms for Examples #4 and #5 are identical, only the ROCs differ
- For causal inputs, the ROC is outside the largest pole of the z-transform
- For anti-causal inputs, the ROC is inside the smallest pole of the z-transform





ROC for Special Sequence Types

- Type 1: Finite Length Sequences

$$x(n) = 0, \qquad n < n_1,\ n > n_2$$

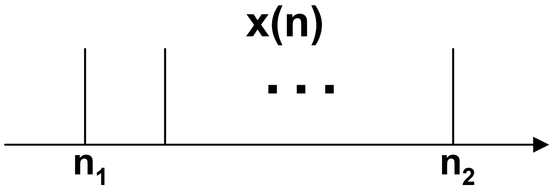
$$X(z) = \sum_{n=n_1}^{n_2} x(n) z^{-n} = x(n_1) z^{-n_1} + \ldots + x(n_2) z^{-n_2}$$

- $X(z)$ converges everywhere except possibly at $z = 0$, and/or $z = \infty$

  -if $n_1 \geq 0$ then ROC contains the point $z = \infty$

  -if $n_2 \leq 0$ then ROC contains the point $z = 0$

- Example: $x(n) = u(n - n_1) - u(n - n_2 - 1),\ n_2 \geq n_1$



ROC for Special Sequence Types

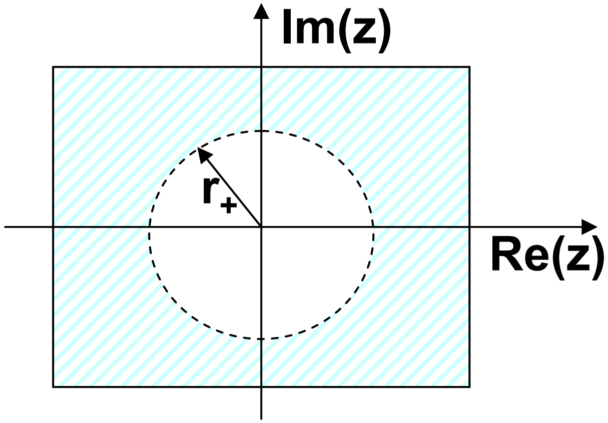
- Type 2: Right-Sided Sequences

$$x(n) = 0, \qquad n \leq n_0$$

$$X(z) = \sum_{n=n_0}^{\infty} x(n) z^{-n}; \text{ ROC: } |z| > |r_+|$$

All poles inside unit circle except possibly at $z = \infty$

If $n_0 \geq 0 \Rightarrow X(z)$ converges at $z = \infty$



- Example: $x(n) = a^n u(n+1),\ a < 1$

- Note: if $x(n) = a^n u(n) \Rightarrow$ first term is gone

  and $X(z)$ converges at $z = \infty$

ROC for Special Sequence Types

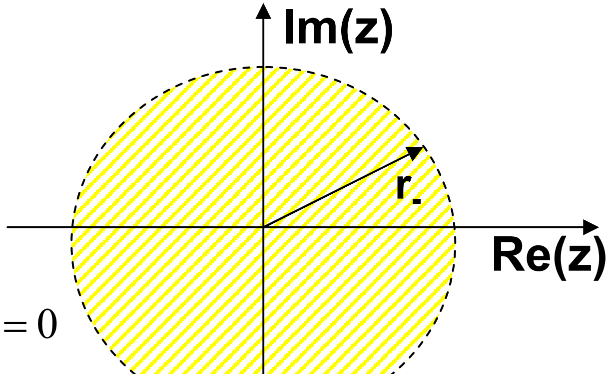
- Type 3: Left-Sided Sequences

$$x(n) = 0, \qquad n > n_0$$

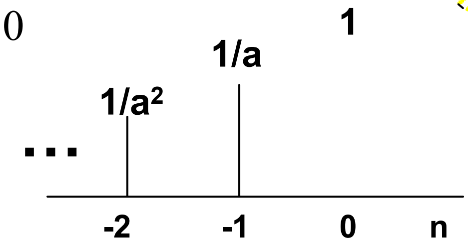
$$X(z) = \sum_{n=-\infty}^{n_0} x(n) z^{-n}; \;\; \text{ROC: } |z| < |r_-|$$

All poles outside unit circle except possibly at $z = 0$

If $n_0 \le 0 \Rightarrow X(z)$ converges at $z = 0$



- Example: $x(n) = a^n u(-n), \quad a > 1$



- Note: for this example $n_0 = 0$ and $X(z)$ converges at $z = 0$

ROC for Special Sequence Types

- Type 4: Two-Sided Sequences

$$\begin{aligned} x(n) &= a^n, \quad n \geq 0 \ \text{(right half)},\ a < 1 \\ &= b^n, \quad n < 0 \ \text{(left half)},\ b > 1 \\ &= a^n u(n) + b^n u(-n-1) \end{aligned}$$

$$X(z) = \sum_{n=-\infty}^{-1} b^n z^{-n} + \sum_{n=0}^{\infty} a^n z^{-n}; \ \text{ROC: } |r_-| < |z| < |r_+|$$

- Example: $x(n) = \left(\frac{1}{2}\right)^n u(n) + 2^n u(-n-1) = 2^{-|n|}$

Since both parts of the summation must converge $\Rightarrow$ ROC:

z-Transform Properties

1. ROC is a ring in the complex z-plane centered at the origin and is of the form: $0 \le r_- < |z| < r_+ \le \infty$
2. The Discrete-time Fourier Transform, DTFT$\{x(n)\}$, of the sequence $x(n)$ converges absolutely iff ROC contains the unit circle in the z-plane
3. The ROC does not contain any poles
4. If $x(n)$ is a finite duration sequence, then the ROC includes the entire z-plane, except possibly the point $z = 0$ or $z = \infty$
5. If $x(n)$ is a right-sided sequence, the ROC is of the form $r_+ < |z|$; the ROC includes the point $z = \infty$ if $x(n) = 0$ for $n < 0$
6. If $x(n)$ is a left-sided sequence, the ROC is of the form $r_- > |z|$; the ROC includes the point $z = 0$ if $x(n) = 0$ for $n > 0$
7. If $x(n)$ is a two-sided sequence, the ROC is of the form $r_- < |z| < r_+$; the ROC is bounded by the poles of the z-transform of $x(n)$
8. The ROC is always a connected region of the z-plane (no gaps or holes)

Notes on ROC

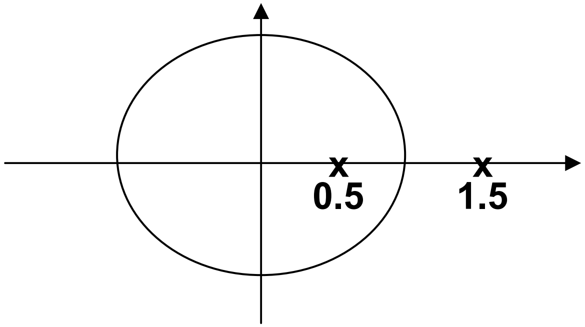
1. Must have ROC with algebraic expression to uniquely specify the z-transform--remember auxiliary conditions with LCCDES
2. Can specify system conditions such as stable system (i.e., DTFT exists $\Rightarrow$ ROC contains the circle $|z|=1$), causal system (i.e., ROC extends out from outermost finite pole), etc., rather than explicitly stating the ROC

EXAMPLE:

stable system $\Rightarrow$ ROC:

causal system $\Rightarrow$ ROC:

anticausal system $\Rightarrow$ ROC:



z-Transform Properties

$x_1(n) \leftrightarrow X_1(z)$ with ROC: R_{x_1}

$x_2(n) \leftrightarrow X_2(z)$ with ROC: R_{x_2}

1. Linearity:

$ax_1(n)+bx_2(n) \leftrightarrow aX_1(z)+bX_2(z)$ with ROC: $\supset R_{x_1} \cap R_{x_2}$

ROC at least the intersection of the individual ROCs; it may be larger if there are pole-zero cancellations in the sum

2. Time-shifting:

$x(n-n_d) \leftrightarrow z^{-n_d} X(z)$ with ROC: R_x plus possible addition or deletion of points $z=0$ or $z=\infty$

If $n_d > 0$, delaying $x(n)$; if $n_d < 0$, advancing $x(n)$; each 1 sample delay (advance) introduces a factor of z^{-1}(or z) into the z-transform

z-Transform Properties

3. Multiplication by an exponential:

$z_0^n x(n) \leftrightarrow X(z/z_0)$ with ROC: $|z_0| R_x \Rightarrow$ all poles/zeros are multiplied by factor $|z_0|$, so ROC is scaled by $|z_0|$

4. Differentiation:

$$nx(n) \leftrightarrow -z\frac{dX(z)}{dz}$$ with ROC: R_x

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$-z\frac{dX(z)}{dz} = -z\sum_{n=-\infty}^{\infty} -nx(n) z^{-n-1}$$

$$= \sum_{n=-\infty}^{\infty} nx(n) z^{-n} = ZT\left\{nx(n)\right\}$$

z-Transform Properties

5. Complex Conjugation:

$x^*(n) \leftrightarrow X^*(z^*)$ with ROC: R_x

$$y(n) = x^*(n)$$

$$Y(z) = \sum_{n=-\infty}^{\infty} x^*(n) z^{-n} = \left[\sum_{n=-\infty}^{\infty} x(n) \left(z^*\right)^{-n} \right]^*$$

$$= \left[X(z^*) \right]^*$$

Note: conjugation commutes, associates, and distributes, i.e.,

$$(a+b)^* = a^* + b^*,\ (ab)^* = a^* b^*,\ (a/b)^* = a^*/b^*,$$

$$\left[a(b+c) \right]^* = a^* \left(b^* + c^* \right)$$

6. Time Reversal:

$x^*(-n) \leftrightarrow X^*(1/z^*)$ with ROC: $1/R_x \Rightarrow$

if $R_x : r_- < |z| < r_+$ then $1/R_x : 1/r_+ < |z| < 1/r_-$

$$y(n) = x^*(-n)$$

$$Y(z) = \sum_{n=-\infty}^{\infty} x^*(-n) z^{-n} = \left[\sum_{m=-\infty}^{\infty} x(m) \left(z^{*-1}\right)^{-m} \right]^*$$

$$= X^*(z^{*-1})^* = X^*(1/z^*)$$

z-Transform Properties

7. Convolution of Sequences;

$$w(n) = x_1(n) * x_2(n)$$

$$W(z) = \sum_{n=-\infty}^{\infty} w(n) z^{-n} = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k) z^{-n}$$ where ROC: R_w

For $z \in R_w$ series is absolutely convergent $\Rightarrow$ can interchange summations, giving:

$$W(z) = \sum_{k=-\infty}^{\infty} x_1(k) \sum_{n=-\infty}^{\infty} x_2(n-k) z^{-n}$$

We recognize that the second term is just $X_2(z) z^{-k}$ from Property #2

$$W(z) = X_2(z) \sum_{k=-\infty}^{\infty} x_1(k) z^{-k}$$

$$= X_1(z) X_2(z)$$ with ROC: $R_w \supset R_{x_1} \cap R_{x_2}$

R_w can be larger than $R_{x_1} \cap R_{x_2}$ if there is any pole-zero cancellation

Example:

z-Transform Properties

8. Initial Value Theorem:

$x(0) = \lim_{z \to \infty} X(z)$ for causal $x(n)$ or anticausal $x(n)$

For causal sequences we have:

since each term in the sum tends to 0 exponentially

The Inverse z-Transform

$$x(n) = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$$ (Cauchy Integral Theorem)

- It is hard to find $x(n)$ by doing the integral directly!
- Instead use one of the following methods of solution:

1. inspection
2. properties of the z-transform
3. partial fraction expansions
4. long division
5. power series expansion

The Inverse z-Transform

1. Inspection:

Can use tables of z-transforms (e.g., Table 3.1 from O&S)

$$a^n u(n) \leftrightarrow \frac{1}{1-az^{-1}}, \quad |z| > |a|$$

$$\therefore X(z) = \frac{1}{1-0.5z^{-1}}, \, |z| > 0.5 \Rightarrow x(n) = (0.5)^n u(n)$$

2. Properties:

Use known relationships to find inverse transform

$$X(z) = \frac{z^{-3}}{1-0.5z^{-1}} + \frac{z^{-1}}{1-z^{-1}}, \quad |z| > 1$$

$$x(n) = (0.5)^{n-3} u(n-3) + u(n-1)$$

The Inverse z-Transform

3. Partial Fraction Expansions:

- Can use PFE to get $X(z)$ in a form suitable for inspection or properties

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{l=0}^{N} a_l z^{-l}}$$

- For this function there will be M zeros and N poles at non-zero locations in the z-plane $\Rightarrow$ can represent $X(z)$ as

$$X(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}\left(1 - c_k z^{-1}\right)}{\prod_{l=1}^{N}\left(1 - d_l z^{-1}\right)},$$

where c_k are the non-zero zeros of $X(z)$ and d_l are the non-zero poles of $X(z)$

- If $M < N \Rightarrow X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}, \quad A_k = \left(1 - d_k z^{-1}\right) X(z)\Big|_{z=d_k}$

The Inverse z-Transform

Example:

The Inverse z-Transform

4. Long Division:

Partial Fraction Expansion (PFE) required that $M < N$. If $M \geq N$, can first perform long division to get $X(z)$ in a form where $M < N$

$$X(z) = \frac{N(z)}{D(z)}, \quad |z| > r$$

- If $M \geq N$, divide until $X(z) = C(z) + \frac{R(z)}{D(z)}, \quad |z| > r$

  where the order of $R(z) < N$

$$X(z) = \frac{a_0 + a_1 z^{-1} + \ldots + a_M z^{-M}}{b_0 + b_1 z^{-1} + \ldots + b_N z^{-N}}$$

$$\begin{array}{r|l} & c_{M-N} z^{-M+N} + \ldots + c_1 z^{-1} + c_0 \\ \hline b_N z^{-N} + \ldots + b_1 z^{-1} + b_0 & a_M z^{-M} + \ldots + a_1 z^{-1} + a_0 \end{array}$$

$R(z)$ is the remainder of the division and has order $< N \Rightarrow$ can use PFE

The Inverse z-Transform

Example:

The Inverse z-Transform

5. Power Series:

If the z-transform is in a power series form $\Rightarrow$ can find the coefficients of $x(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \ldots + x(-1)z + x(0) + x(1)z^{-1} + \ldots$$

Example: $X(z) = e^{a/z},\ |z| > 0$ (causal sequence)

$$X(z) = \sum_{n=0}^{\infty} \frac{(a/z)^n}{n!} = \sum_{n=0}^{\infty} \frac{a^n}{n!} z^{-n}$$

$$x(n) = \frac{a^n}{n!} u(n)$$

The Inverse z-Transform

Example: $X(z) = \dfrac{1}{1 - az^{-1}}, \ |z| > |a|, \ M = 0, \ N = 1$