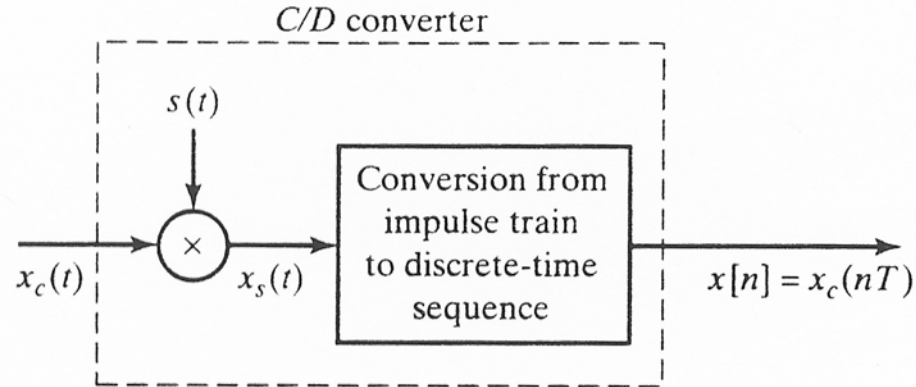
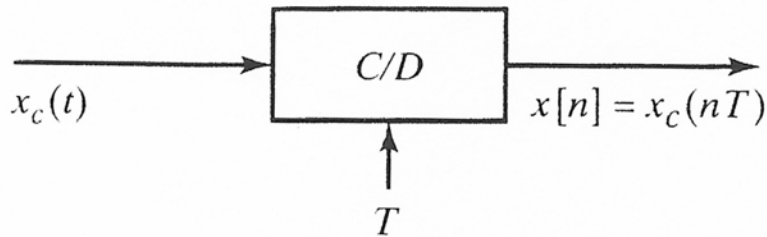


Discrete-Time Signal Processing

Lecture 11

Continuous Time \rightarrow Discrete Time \rightarrow
Continuous Time Systems

Continuous-Discrete



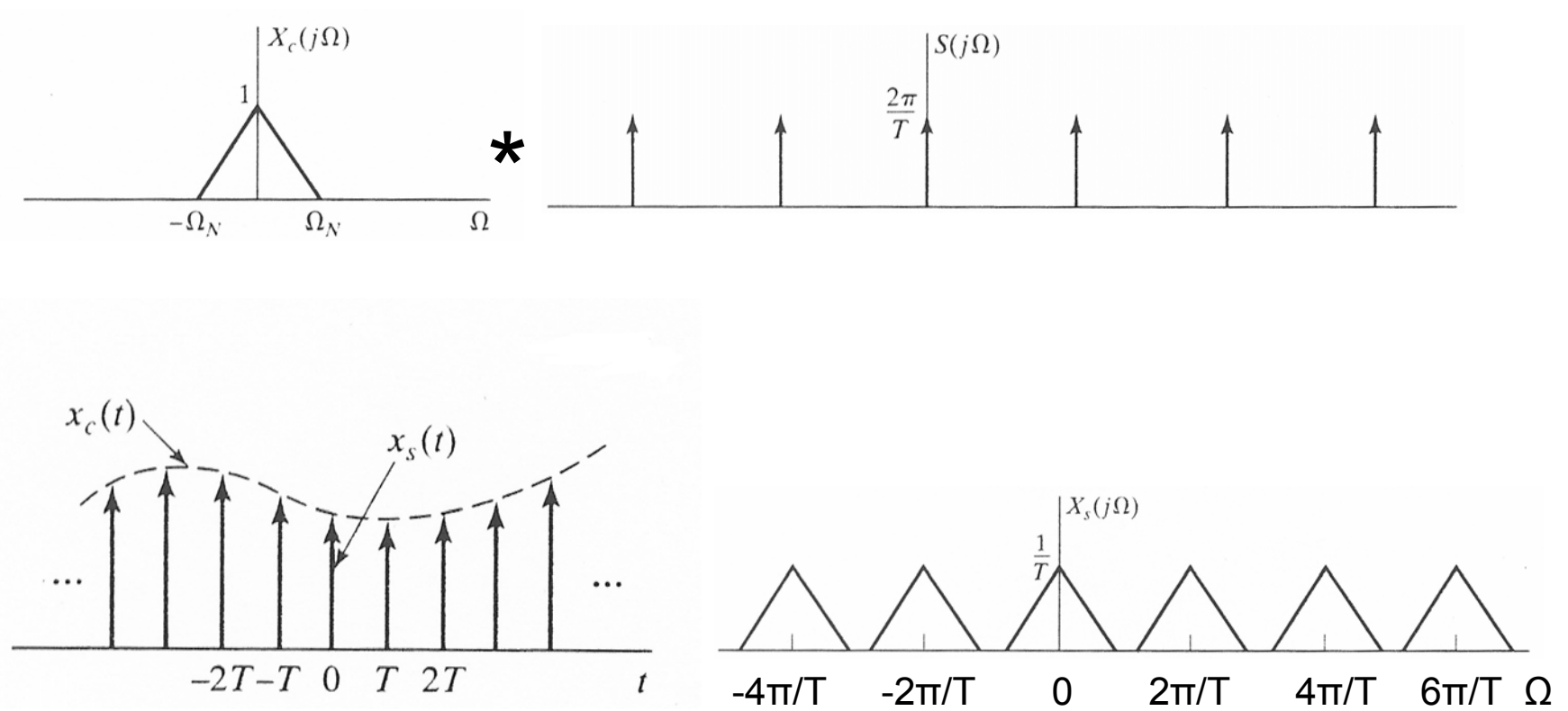
$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \xleftrightarrow{FT} S(j\Omega) =$$

$$x_s(t) = x_c(t) s(t) \longleftrightarrow X_s(j\Omega) =$$

$$= x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) =$$

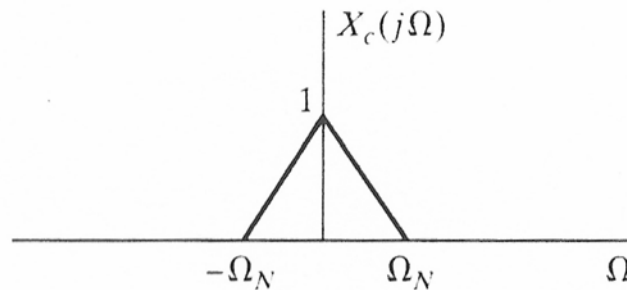
$$= \sum_{n=-\infty}^{\infty} x_c(nT) \delta(t - nT) =$$

Continuous-Discrete

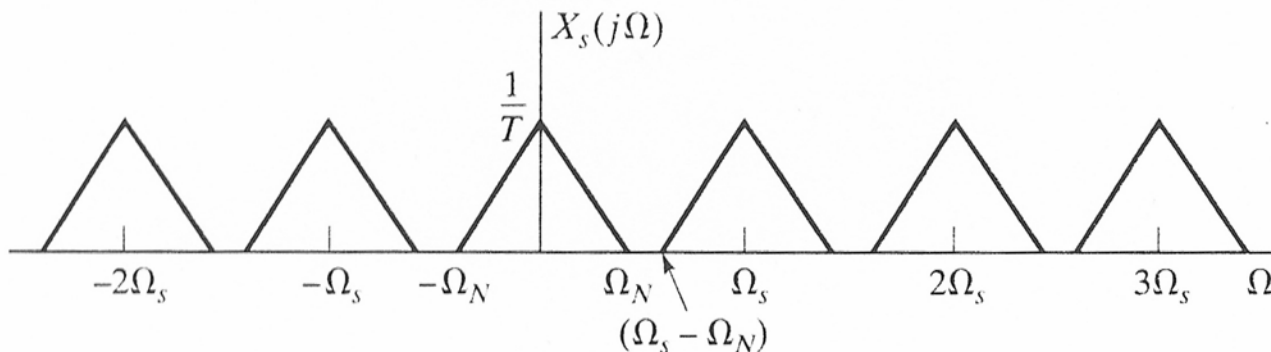


Continuous-Discrete

- If $X_c(j\Omega) = 0 \quad |\Omega| > \Omega_N$
 \Rightarrow no aliasing as long as

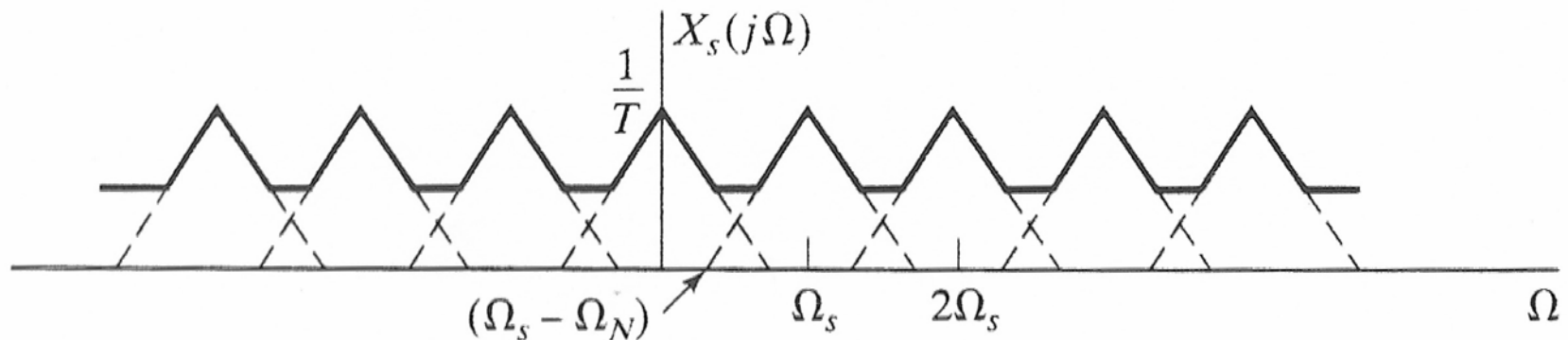


- If $\Omega_s = \frac{2\pi}{T} \geq 2\Omega_N$



Continuous-Discrete

- If $\Omega_s = \frac{2\pi}{T} < 2\Omega_N$



- Aliasing occurs \Rightarrow cannot recover $x_c(t)$ from $x_s(t)$
- No aliasing occurs \Rightarrow can recover $x_c(t)$ from $x_s(t)$ via lowpass filtering out all copies of spectrum--except the one centered at 0 frequency with cutoff $\Omega_N < \Omega_c < \Omega_s - \Omega_N$

Continuous-Discrete Example

$$x_c(t) = \cos(\Omega_0 t) \xleftrightarrow{FT} X_c(j\Omega) = \pi\delta(\Omega - \Omega_0) + \pi\delta(\Omega + \Omega_0)$$

Nyquist Sampling Theorem

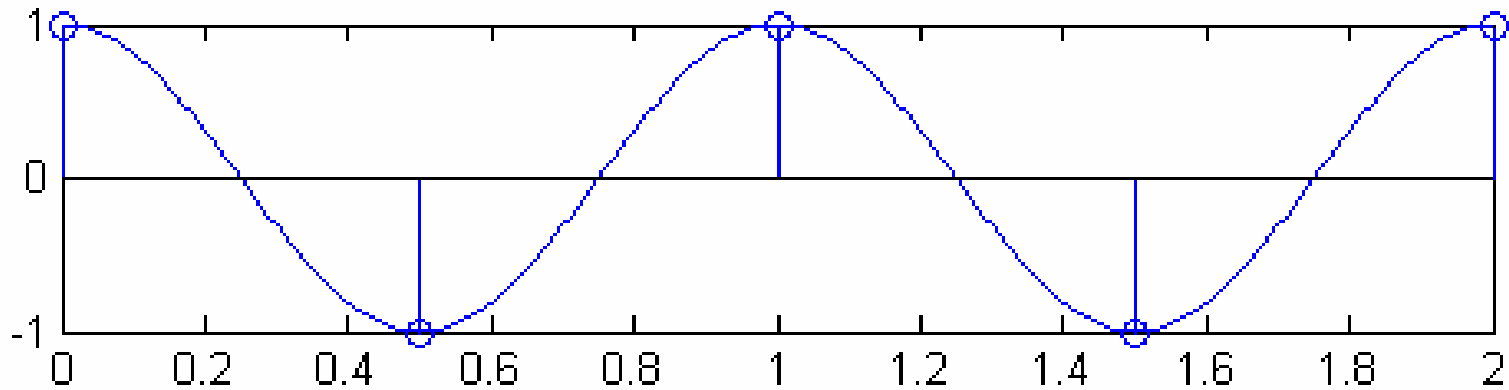
- If $X_c(j\Omega) = 0$, $|\Omega| > \Omega_N$ then $x_c(t)$ can be uniquely reconstructed from $x(n) = x_c(nT)$,

$$\forall n \text{ if } \boxed{\Omega_s = \frac{2\pi}{T} > 2\Omega_N}$$

- Ω_N is called the Nyquist frequency
- $2\Omega_N$ is called the Nyquist rate
- Ω_s is called the sampling rate

Aliasing in the Time Domain

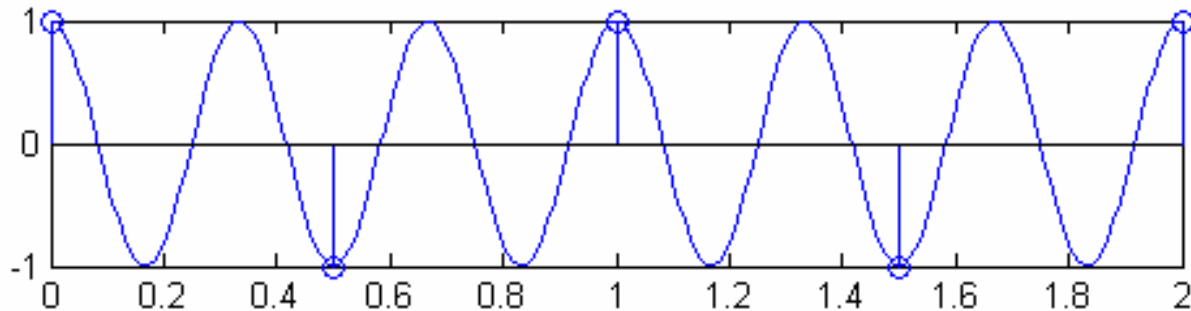
$$x_c(t) = \cos(2\pi t), \quad \Omega_0 = 2\pi, \quad \Omega_s = 4\pi$$



$\Omega_s = 2\Omega_0 \Rightarrow$ can obtain $x_c(t)$ from the samples $x_s(t)$

Aliasing in the Time Domain

$$x'_c(t) = \cos(6\pi t), \quad \Omega_0 = 6\pi, \quad \Omega_s = 4\pi$$



$\Omega_s < 2\Omega_0 \Rightarrow$ will reconstruct $x'_s(t)$ as

$$x'_r(t) = \cos((\Omega_s - \Omega_0)t)$$

$$= \cos((-2\pi)t) = \cos(2\pi t) = x_c(t)$$

$x'_s(t) \equiv x_s(t) \Rightarrow$ we cannot distinguish $x'_c(t)$
from $x'_s(t)$ because we did not sample $x'_c(t)$
fast enough (Ω_s too small)

Relation Between DT and CT Signals

- We need to connect $x(n)$ to $x_s(t)$, or alternatively, $X(e^{j\omega})$ to $X_s(j\Omega)$:

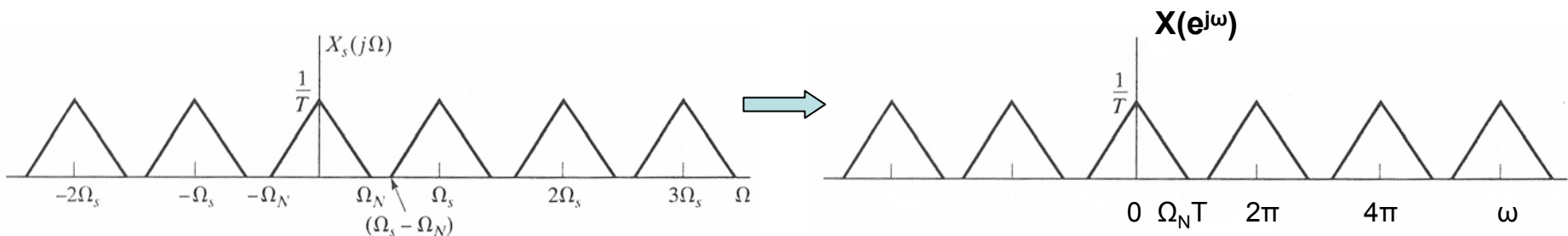
$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT) e^{-j\Omega T n}$$

Relation Between DT and CT Signals

$$X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c \left[j \left(\frac{\omega}{T} - \frac{2\pi k}{T} \right) \right]$$

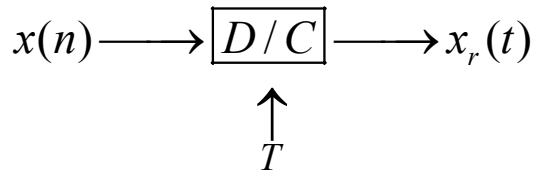
$\Rightarrow X(e^{j\omega})$ is a frequency-scaled version of $X_s(j\Omega)$ where $\Omega_s \rightarrow 2\pi$

$\Rightarrow X(e^{j\omega})$ and consequently $x(n)$ has no inherent notion of the sampling period T (or $\Omega_s = 2\pi / T$)

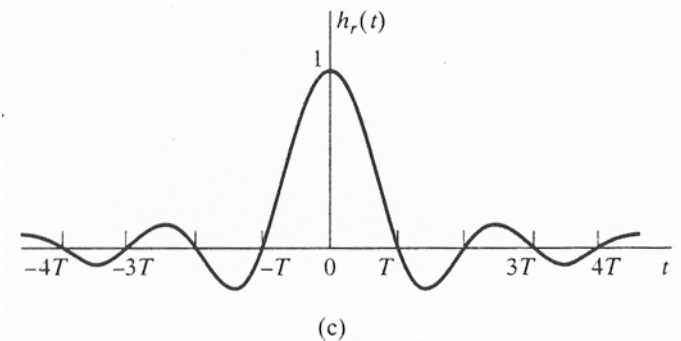
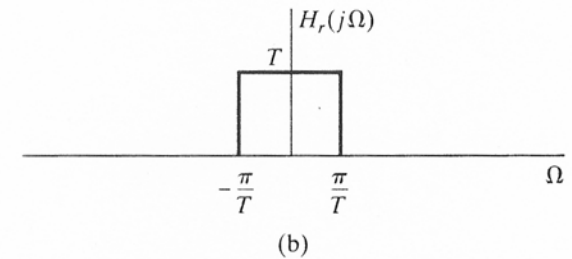
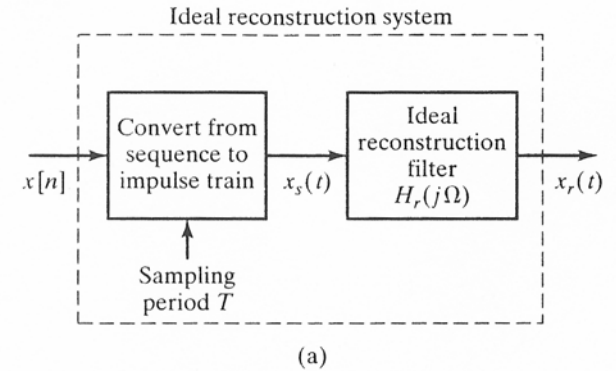


DT \rightarrow CT Conversion

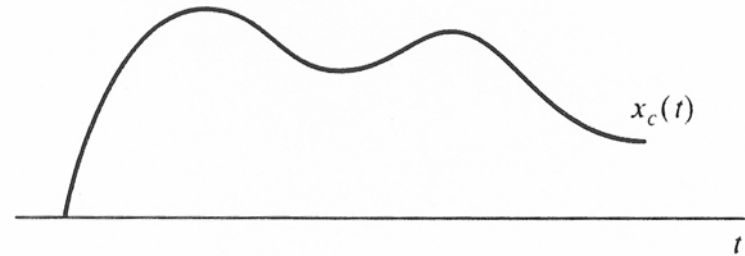
- Let's now look at the discrete-continuous conversion process:



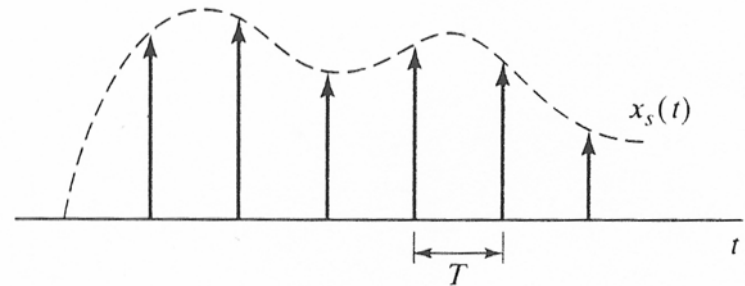
- Note that $h_r(0) = 1$ and $h_r(nT) = 0 \quad \forall n \neq 0$



DT \rightarrow CT Conversion

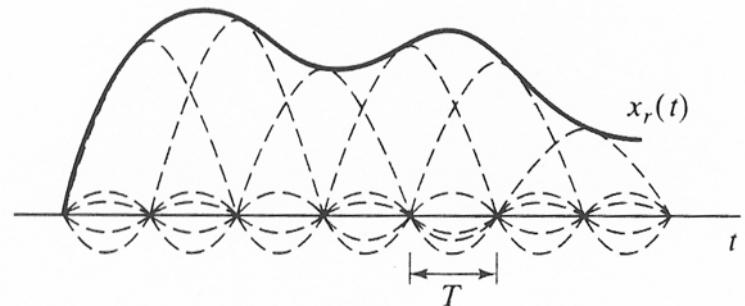


(a)



(b)

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(n) \frac{\sin \left[\pi (t - nT) / T \right]}{\left[\pi (t - nT) / T \right]}$$



(c)

Bandlimited Interpolation

- The filter $h_r(t)$ interpolates between the known samples at locations $t = nT$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(n)h_r(t - nT)$$

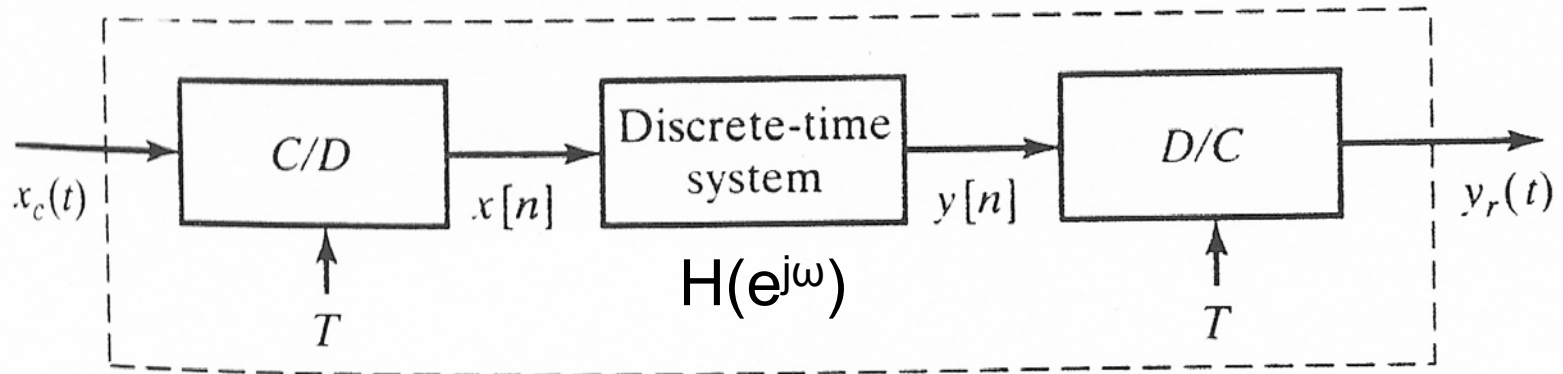
- In the frequency domain we get

$$X_r(j\Omega) =$$

Bandlimited Interpolation

- Remember that through the C/D process we get: $\Omega_N \longrightarrow \omega_0 = \Omega_N T$
- Through the D/C process we get: $\omega_0 = \Omega_N T \longrightarrow \omega_0 / T = \Omega_N$

DT Processing of CT Signals



- Only the D/C subsystem is LTI; the DT system is often LTI; the C/D subsystem is not LTI \Rightarrow overall system is generally not LTI

DT Processing of CT Signals

- If the *input is bandlimited* ($X_c(j\Omega) = 0, |\Omega| < \pi/T$) and the *DT system is LTI* \Rightarrow system behaves like an LTI system

DT Processing of CT Signals

- Thus if $X_c(j\Omega)$ is bandlimited and the sampling rate is above the Nyquist rate, the output is related to the input through an equation of the form:

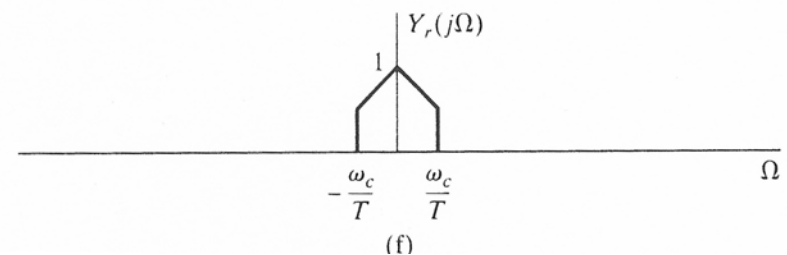
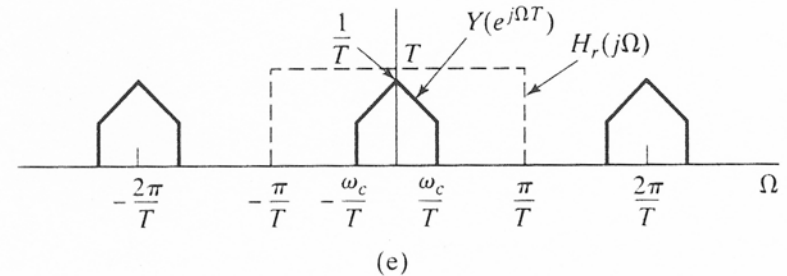
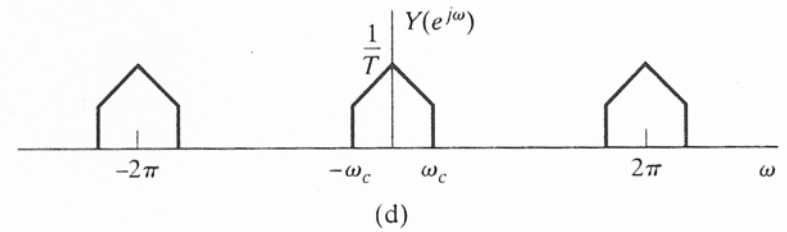
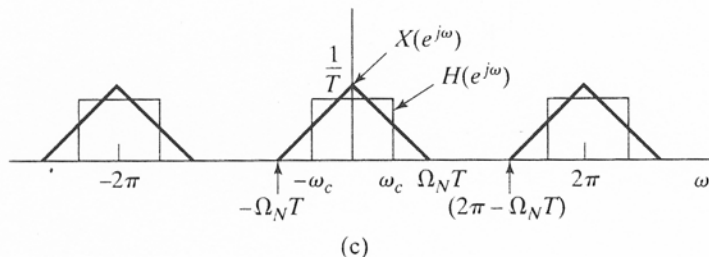
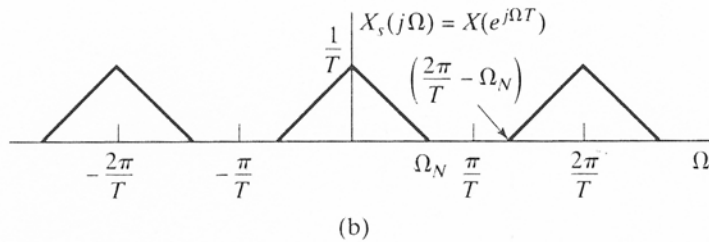
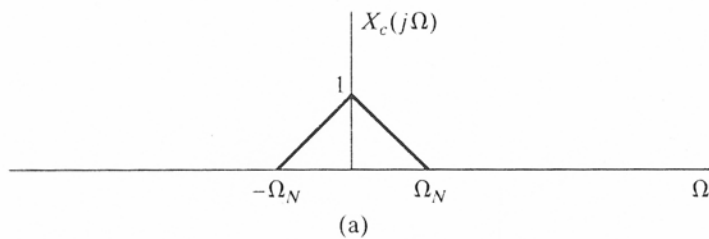
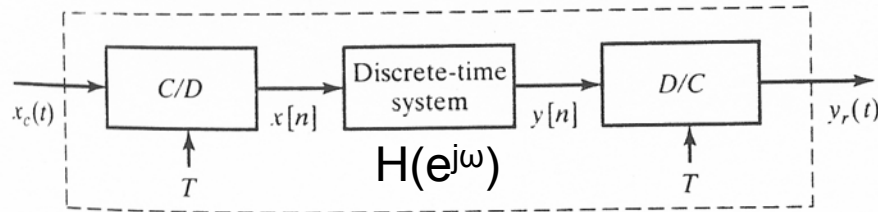
$$Y_r(j\Omega) = H_c(j\Omega) \cdot X_c(j\Omega)$$

- where

$$H_c(j\Omega) = \begin{cases} H(e^{j\Omega T}) & |\Omega| < \pi / T \\ 0 & \text{otherwise} \end{cases}$$

- Thus the overall continuous-time system is equivalent to an LTI system whose effective frequency response is given by $H_c(j\Omega)$
- This analysis may be valid if some aliasing occurs in the C/D subsystem so long as $H(e^{j\omega})$ does not pass the aliased components

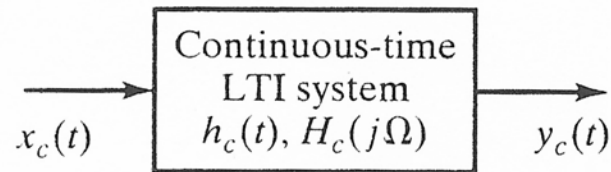
DT Processing of CT Signals



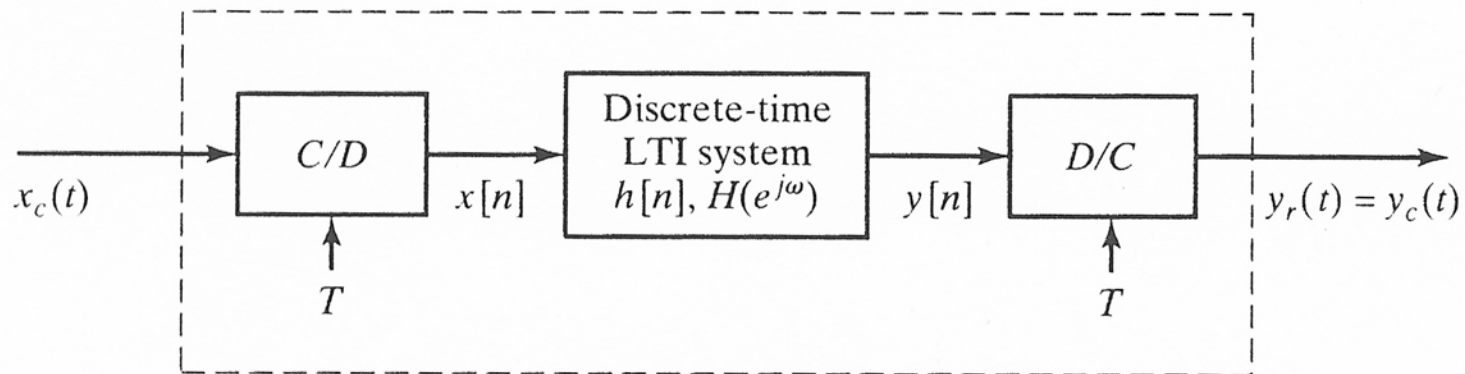
- a) Fourier transform of bandlimited input signal; b) Fourier transform of sampled input versus Ω ; c) Fourier transform $X(e^{j\omega})$ of sequence of samples and frequency response $H(e^{j\omega})$ of discrete time system versus ω ; d) Fourier transform of output of discrete-time system; e) Fourier transform of output of discrete-time system and frequency response of ideal reconstruction filter versus Ω ; f) Fourier transform of output

DT Processing of CT Signals

- Given that the input $x_c(t)$ is bandlimited to π/T , we obtain the following equivalence:



(a)



$$H_{\text{eff}}(j\Omega) = H_c(j\Omega)$$

(b)

DT Processing of CT Signals

- If we know the desired CT LTI system ($h_c(t)$ or $H_c(j\Omega)$), we can find the equivalent DT system ($h(n)$ or $H(e^{j\omega})$):

$$H(e^{j\omega}) = H_c(j\omega/T) \quad |\omega| < \pi$$

- where T is chosen such that $H_c(j\Omega) = 0$, $|\Omega| \geq \pi/T$

$$\Rightarrow h(n) = Th_c(nT)$$

- \Rightarrow Given the desired CT system response, we can find the equivalent DT system

DT Processing of CT Signals

- Example: Suppose we want to implement an ideal LPF

$$H_c(j\Omega) = \begin{cases} 1 & |\Omega| \leq \Omega_c \\ 0 & \text{otherwise} \end{cases}$$

$$h_c(t) = \frac{\sin(\Omega_c t)}{\pi t}$$

- Therefore we have to design a DT filter with the specs:

- The Fourier Transform of $h(n)$ is

$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \text{otherwise} \end{cases}$$

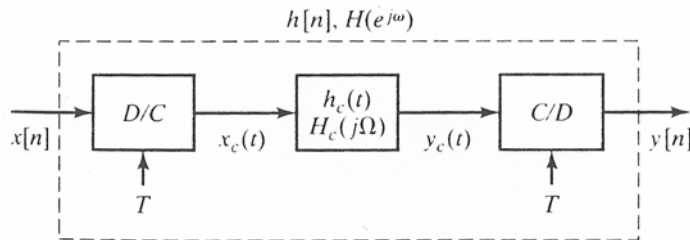
- We can easily see that

CT Processing of DT Signals

- Basic DT processing model:

$$x(n) \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow y(n)$$

- We want to implement this system as:



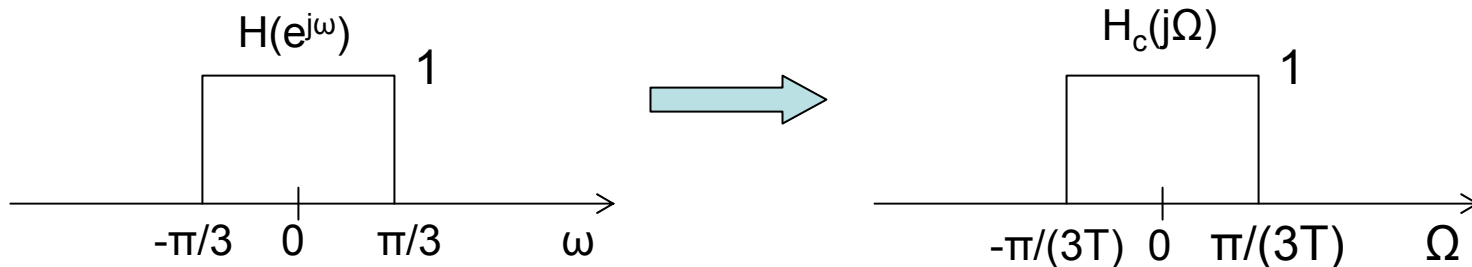
- Note that there is no aliasing for any choice of T .
- This system is LTI, if $H_c(j\Omega)$ is LTI.
- System relationships:

$$x(n) = x_c(nT); \quad y(n) = y_c(nT)$$

$$x_c(t) = \sum_{n=-\infty}^{\infty} x(n) \frac{\sin\left[\pi(t-nT)/T\right]}{\pi(t-nT)/T}$$

CT Processing of DT Signals

- Frequency domain relationships for system:



CT Processing of DT Signals

- Example: Non-integer delays

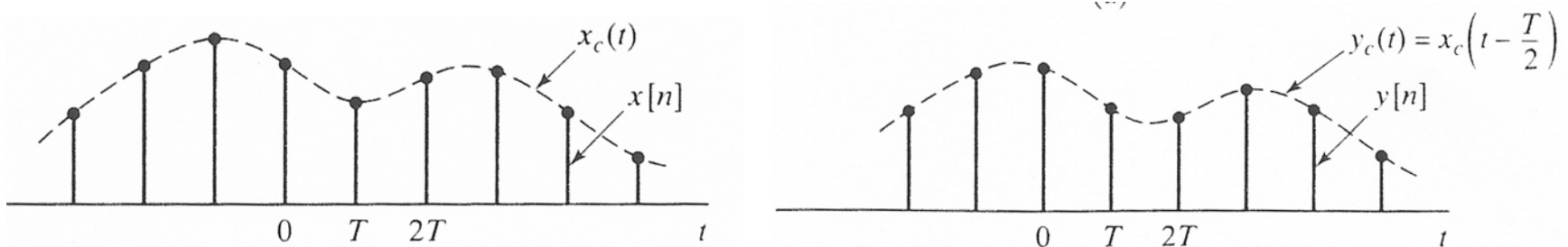
$$H(e^{j\omega}) = e^{-j\omega\alpha} \quad |\omega| < \pi$$

$$GD\{H(e^{j\omega})\} = \alpha$$

$$H_c(j\Omega) =$$

- We can define $H_c(j\Omega)$ everywhere because it doesn't matter what $H_c(j\Omega)$ is outside $|\Omega| > \pi T$

⇒ bandlimited interpolation of $x(n)$ as shown below



“Half-sample” delay leads to bandlimited interpolation of input signal

CT Processing of DT Signals

- Direct convolution system representation obtained as:

$$\begin{aligned}y(n) &= y_c(nT) = x_c(nT - \alpha T) \\&= \sum_{k=-\infty}^{\infty} x(k) \frac{\sin[\pi(t - \alpha T - kT)/T]}{\pi(t - \alpha T - kT)/T} \Bigg|_{t=nT} \\&= \sum_{k=-\infty}^{\infty} x(k) \frac{\sin[\pi(n - k - \alpha)]}{\pi(n - k - \alpha)} \\&= x(n) * \frac{\sin[\pi(n - \alpha)]}{\pi(n - \alpha)} = x(n) * h(n)\end{aligned}$$

- When α is an integer $\Rightarrow h(n) = \delta(n - \alpha)$

$$\text{otherwise, } h(n) = \frac{\sin[\pi(n - \alpha)]}{\pi(n - \alpha)}, \quad -\infty < n < \infty \Rightarrow \text{IIR filter}$$

- Note that $h(n)$ may not be symmetric, but $H(e^{j\omega})$ still has linear phase.